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# THE VARIATION PROBLEM IN GENERALIZED LAGRANGE-HAMILTON SPACES

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**Abstract.** Many significant geometers have contributed to the generalization of Riemann spaces in different directions. In this way arise Finsler spaces, Lagrange spaces, Hamilton spaces, $k$-Lagrange and $k$-Hamilton spaces, Lagrange spaces of order $k$ and Hamilton spaces of order $k$. In references [1–19] an incomplete selection of papers and books connected with these spaces is given. In all these spaces the variation problem is solved. Here, this problem is examined in generalized Lagrange-Hamilton spaces, $(GLH)^{(nk)}$, introduced in [9]. All the spaces mentioned above appear as special cases of $(GLH)^{(nk)}$.

In the first section, the group of coordinates transformation is given and the natural bases $\bar{B}$ and $\bar{B}^*$ of tangent and cotangent spaces $T(GLH)^{(nk)}$ and $T^*(GLH)^{(nk)}$ are examined.

In the second section, the solution of the variation problem of the integral of action for the extreme value of the fundamental function $F(x, y^1, \ldots, y^k, p_1, \ldots, p_k)$ is obtained. Here, the modified Liouville vectors $I_A(v, h)$ are applied. The connection between notations used here and in [13–15] can be easily established. The generalized Euler-Lagrange (E-L) equations in $(GLH)^{(nk)}$ reduce to the known (E-L) equations in generalized Lagrange spaces.

In the third section, the generalizations of Craig-Synge covectors are given and some important theorems connected with this problem in $(GLH)^{(nk)}$ are proved. The method of proofs is the same as in [13].

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## 1. Group of transformations, tangent and cotangent spaces

Generalized Lagrange-Hamilton spaces are introduced in [9]. We shall recall only the basic notions which are necessary for understanding the variation problem in these spaces.

Let us denote by $(LH)^{(nk)}$ the $(2k+1)n$ dimensional $C^\infty$ manifold in which a point $(y, p) = (x = y^{(0)}, y^{(1)}, y^{(2)}, \ldots, y^{(k)}, p_{(1)}, p_{(2)}, \ldots, p_{(k)})$ has the coordinates

$$(x^a = y^{0a}, y^{1a}, y^{2a}, \ldots, y^{ka}, p_{1a}, p_{2a}, \ldots, p_{ka}), \quad a = \overline{1, n}.$$

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Some curve $c \in (LH)^{(nk)}$ is given by $c : t \in [a,b] \to c(t) \in (LH)^{(nk)}$. A point $(y,p) \in c(t)$ has the coordinates

$$(x^a(t) = y^{0a}(t), y^{1a}(t), \dots, y^{ka}(t), p_{1a}(t), \dots, p_{ka}(t)),$$

where

$$y^{Aa}(t) = d_t^A y^{0a}(t) \quad A = \overline{1,k}, \quad d_t^A = \frac{d^A}{dt^A}, \tag{1.1}$$

$$p_{\alpha a}(t) = d_t^{\alpha-1} p_{1a}(t), \quad \alpha = \overline{1,k}, \quad d_t^{\alpha-1} = \frac{d^{\alpha-1}}{dt^{\alpha-1}}.$$

The allowable coordinate transformations are given by

$$x^{a'} = x^{a'}(x^a) \Leftrightarrow x^a = x^a(x^{a'}) \tag{1.2}$$

$$\begin{aligned}
y^{1a'} &= B_a^{a'} y^{1a}, \quad B_a^{a'} = \partial_{0a} x^{a'} = \partial_a x^{a'},\\
\partial_{Aa} &= \frac{\partial}{\partial y^{Aa}} \quad A = \overline{0,k}, \quad rank(B_a^{a'}) = n, \dots,\\
y^{Aa'} &= \binom{A-1}{0} (d_t^{A-1} B_a^{a'}) y^{1a} + \binom{A-1}{1} (d_t^{A-2} B_a^{a'}) y^{2a} + \cdots\\
&\quad \cdots + \binom{A-1}{A-1} B_a^{a'} y^{Aa} = d_t^{A-1}(B_a^{a'} y^{1a}), \dots,\\
y^{ka'} &= \binom{k-1}{0} (d_t^{k-1} B_a^{a'}) y^{1a} + \binom{k-1}{1} (d_t^{k-2} B_a^{a'}) y^{2a} + \cdots\\
&\quad \cdots + \binom{k-1}{k-1} B_a^{a'} y^{ka} = d_t^{k-1}(B_a^{a'} y^{1a}),\\
p_{1a'} &= B_{a'}^a p_{1a} \quad B_{a'}^a = \partial_{0a'} x^a = \frac{\partial x^a}{\partial x^{a'}} = B_{a'}^a(t), \dots,\\
p_{\alpha a'} &= \binom{\alpha-1}{0} (d_t^{\alpha-1} B_{a'}^a) p_{1a} + \binom{\alpha-1}{1} (d_t^{\alpha-2} B_{a'}^a) p_{2a} + \cdots\\
&\quad \cdots + \binom{\alpha-1}{\alpha-1} B_{a'}^a p_{\alpha a}, \dots,\\
p_{ka'} &= \binom{k-1}{0} (d_t^{k-1} B_{a'}^a) p_{1a} + \binom{k-1}{1} (d_t^{k-2} B_{a'}^a) p_{2a} + \cdots\\
&\quad \cdots + \binom{k-1}{k-1} B_{a'}^a p_{ka}.
\end{aligned}$$

**Theorem 1.1.** *The transformations of type* (1.2) *on the common domain form a group.*

**Definition 1.1.** *The generalized Lagrange-Hamilton space* $(GLH)^{(nk)}$ *of order* $k$ *is a* $(LH)^{(nk)}$ *space, where the group of allowable transformations is given by* (1.2)*, and in which a fundamental function*

$$F(x, y^{(1)}, y^{(2)}, \dots, y^{(k)}, p_{(1)}, p_{(2)}, \dots, p_{(k)})$$

*is given, where* $F : U \to R$ *is differentiable on* $\tilde{U}$ *(rank* $[y^{1a}] = 1$*, rank* $[p_{1a}] = 1$*) and continuous at those points of* $U$*, where* $y^{1a}$ *and* $p_{1a}$ *are equal to zero,* $U$ *is a domain in* $(GLH)^{(nk)}$*.*

The natural basis, $\bar{B}_{LH}$ of $T(GLH)^{(nk)}$, as usual, consists of partial derivatives of variables, i.e.

$$\bar{B}_{LH} = \{\partial_{0a}, \partial_{1a}, \dots, \partial_{ka}, \partial^{1a}, \partial^{2a}, \dots, \partial^{ka}\}, \tag{1.3}$$

$$\partial_{0a} = \partial_a = \frac{\partial}{\partial x^a} = \frac{\partial}{\partial y^{0a}}, \quad \partial_{Aa} = \frac{\partial}{\partial y^{Aa}}\ A = \overline{1,k}, \quad \partial^{\alpha a} = \frac{\partial}{\partial p_{\alpha a}}, \ \alpha = \overline{1,k}.$$

**Theorem 1.2.** *The elements of* $\bar{B}_{LH}$ *transform in the following way:*

(1.4)

$$\begin{aligned}
\partial_{0a} = {} & (\partial_{0a}y^{0a'})\partial_{0a'} + (\partial_{0a}y^{1a'})\partial_{1a'} + (\partial_{0a}y^{2a'})\partial_{2a'} + (\partial_{0a}y^{3a'})\partial_{3a'} + \cdots + (\partial_{0a}y^{ka'})\partial_{ka'} \\
& + (\partial_{0a}p_{1a'})\partial^{1a'} + (\partial_{0a}p_{2a'})\partial^{2a'} + (\partial_{0a}p_{3a'})\partial^{3a'} + \cdots + (\partial_{0a}p_{ka'})\partial^{ka'}, \\
\partial_{1a} = {} & (\partial_{1a}y^{1a'})\partial_{1a'} + (\partial_{1a}y^{2a'})\partial_{2a'} + (\partial_{1a}y^{3a'})\partial_{3a'} + \cdots + (\partial_{1a}y^{ka'})\partial_{ka'} \\
& + (\partial_{1a}p_{2a'})\partial^{2a'} + (\partial_{1a}p_{3a'})\partial^{3a'} + \cdots + (\partial_{1a}p_{ka'})\partial^{ka'}, \dots \\
\partial_{ka} = {} & (\partial_{ka}y^{ka'})\partial_{ka'} \\
\partial^{1a} = {} & (\partial^{1a}p_{1a'})\partial^{1a'} + (\partial^{1a}p_{2a'})\partial^{2a'} + (\partial^{1a}p_{3a'})\partial^{3a'} + \cdots + (\partial^{1a}p_{ka'})\partial^{ka'}, \\
\partial^{2a} = {} & (\partial^{2a}p_{2a'})\partial^{2a'} + (\partial^{2a}p_{3a'})\partial^{3a'} + \cdots + (\partial^{2a}p_{ka'})\partial^{ka'}, \dots, \\
\partial^{ka} = {} & (\partial^{ka}p_{ka'})\partial^{ka'}.
\end{aligned}$$

The natural basis of $T^*(GLH)^{(nk)}$ is

$$\bar{B}^*_{LH} = \{dy^{0a}, dy^{1a}, \dots, dy^{ka}, dp_{1a}, dp_{2a}, \dots, dp_{ka}\}.$$

**Theorem 1.3.** *The elements of* $\bar{B}^*_{LH}$ *transform in the following way:*

(1.5)

$$\begin{aligned}
dy^{0a'} &= (\partial_{0a}y^{0a'})dy^{0a} \\
dy^{1a'} &= (\partial_{0a}y^{1a'})dy^{0a} + (\partial_{1a}y^{1a'})dy^{1a}, \dots, \\
dy^{ka'} &= (\partial_{0a}y^{ka'})dy^{0a} + (\partial_{1a}y^{ka'})dy^{1a} + \cdots + (\partial_{ka}y^{ka'})dy^{ka}, \\
dp_{1a'} &= (\partial_{0a}p_{1a'})dy^{0a} + (\partial^{1a}p_{1a'})dp_{1a},
\end{aligned}$$

$$dp_{2a'} = (\partial_{0a}p_{2a'})dy^{0a} + (\partial_{1a}p_{2a'})dy^{1a} + (\partial^{1a}p_{2a'})dp_{1a} + (\partial^{2a}p_{2a'})dp_{2a}, \dots,$$

$$dp_{ka'} = (\partial_{0a}p_{ka'})dy^{0a} + (\partial_{1a}p_{ka'})dy^{1a} + \dots + (\partial_{(k-1)a}p_{ka'})dy^{(k-1)a} +$$
$$(\partial^{1a}p_{ka'})dp_{1a} + \dots + (\partial^{ka}p_{ka'})dp_{ka}.$$

It is obvious that the elements of $\bar{B}_{LH}$ and $\bar{B}^*_{LH}$ are not transforming as tensors (except for $\partial_{ka}$, $\partial^{ka}$ and $dy^{0a}$). Using the $J$ structure in [9], special adapted bases $B_{LH}$ and $\bar{B}^*_{LH}$ are constructed, such that their elements are tensors. Here, these bases will not be used, so their construction is omitted. For the further application we shall define the special Lagrange-Hamilton $(SLH)^{(nk)}$ spaces by

**Definition 1.2.** *The $(SLH)^{(nk)}$ are such $(LH)^{(nk)}$ spaces in which the group of transformation is reduced to a linear group, i.e. elements of the matrix $(B^{a'}_a)$ are real numbers.*

From Definition 1.2 and (1.2) it follows that in $(SLH)^{(nk)}$ the group of transformation is given by:

$$y^{0a'} = B^{a'}_a y^{0a}, y^{1a'} = B^{a'}_a y^{1a}, \dots, y^{ka'} = B^{a'}_a y^{ka}, \tag{1.6}$$
$$p_{1a'} = B^a_{a'} p_{1a}, \dots, p_{ka'} = B^a_{a'} p_{ka}.$$

From (1.6) it follows that in $(SLH)^{(nk)}$ the elements of $\bar{B}_{SLH}$ and $\bar{B}^*_{SLH}$ are the same as the corresponding elements of $\bar{B}_{LH}$ and $\bar{B}^*_{LH}$. But, their elements are transforming as tensors, namely from (1.4) and (1.5) it follows

$$\partial_{0a} = B^{a'}_a \partial_{0a'}, \dots, \partial_{ka} = B^{a'}_a \partial_{ka'}, B^{a'}_a = \partial_{0a} y^{0a'} \tag{1.7}$$
$$\partial^{1a} = B^a_{a'} \partial^{1a'}, \dots, \partial^{ka} = B^a_{a'} \partial^{ka'}$$
$$dy^{0a'} = B^{a'}_a dy^{0a}, \dots, dy^{ka'} = B^{a'}_a dy^{ka},$$
$$dp_{1a'} = B^a_{a'} dp_{1a}, \dots, dp_{ka'} = B^a_{a'} dp_{ka}.$$

## 2. The variation problem in $(GLH)^{(nk)}$

Let us consider the differentiable curve

$$c^* : t \in [0,1] \to c^*(t) \subset U \subset (GLH)^{(nk)}$$

$U$ is an open set and

$$c^*(t) = r(t) = y^{0a}(t)\partial_{0a} + y^{1a}(t)\partial_{1a} + \cdots$$
$$\cdots + y^{ka}(t)\partial_{ka} + p_{1a}(t)\partial^{1a} + \dots + p_{ka}(t)\partial^{ka},$$

$$y^{Aa}(t) = d^A_t y^{0a}(t), \quad A = \overline{1,k}, \qquad p_{\alpha a}(t) = d^{\alpha-1}_t p_{1a}(t), \quad \alpha = \overline{2,k}.$$

The integral of action $I_{c^*}$ for the fundamental function

$$F(y^0, y^1, \dots, y^k, p_1, \dots, p_k)$$

is given by

$$I_{c^*} = \int_0^1 F(y^{0a}(t), y^{1a}(t), \dots, y^{ka}(t), p_{1a}(t), \dots, p_{ka}(t))dt. \tag{2.1}$$

The curve $c^*_\varepsilon(t) = r(t) + \varepsilon\delta r(t)$ is given by $c^*_\varepsilon : t \in [0,1] \to c^*_\varepsilon(t) \subset U \subset (GLH)^{(nk)}$, where for

$$\delta r(t) = v^{0a}(t)\partial_{0a} + v^{1a}(t)\partial_{1a} + \dots + v^{ka}(t)\partial_{ka} + h_{1a}(t)\partial^{1a} + \dots + h_{ka}(t)\partial^{ka} \tag{2.2}$$

the following relations are valid:

$$v^{Aa}(t) = d_t^A v^{0a}(t), \quad A = \overline{1,k}, \qquad h_{\alpha a}(t) = d_t^{\alpha-1} h_{1a}(t), \quad \alpha = \overline{2,k}. \tag{2.3}$$

We shall suppose that the curves $c^*_\varepsilon(t)$ for every small enough $\varepsilon$ (positive or negative) such that $Imc^*_\varepsilon \subset U$, have the same endpoint and initial point as the curve $c^*(t)$, i.e.

$$c^*_\varepsilon(0) = c^*(0), \qquad c^*_\varepsilon(1) = c^*(1).$$

This will be satisfied if

$$v^{Aa}(0) = v^{Aa}(1) = 0, \quad A = \overline{1,k} \qquad h_{\alpha a}(0) = h_{\alpha a}(1) = 0, \quad \alpha = \overline{2,k}. \tag{2.4}$$

The integral of action $I_{c^*_\varepsilon}$ of $F$ is

$$I_{c^*_\varepsilon} = \int_0^1 F(y^{0a}(t) + \varepsilon v^{0a}(t), \dots, y^{ka}(t) + \varepsilon v^{ka}(t), p_{1a}(t) + \varepsilon h_{1a}(t), \dots, p_{ka} + \varepsilon h_{ka}(t))dt. \tag{2.5}$$

Using Taylor's formula we get

$$I_{c^*_\varepsilon} - I_{c^*} = \delta I + \delta^2 I + \varepsilon^3 R_3, \tag{2.6}$$

where

$$\begin{aligned} \delta I &= \int_0^1 dF dt \\ &= \varepsilon \int_0^1 (v^{0a}\partial_{0a} + v^{1a}\partial_{1a} + \dots + v^{ka}\partial_{ka} + h_{1a}\partial^{1a} + \dots + h_{ka}\partial^{ka})F dt, \end{aligned} \tag{2.7}$$

$$\delta^2 I = \frac{1}{2}\int_0^1 d^2 F dt$$

$$= \frac{\varepsilon^2}{2}\int_0^1 [v^{0a}\partial_{0a} + v^{1a}\partial_{1a} + \cdots + v^{ka}\partial_{ka} + h_{1a}\partial^{1a} + \cdots + h_{ka}\partial^{ka}]^2 F dt.$$

As $\varepsilon$ may be a positive or negative small number, so the necessary condition that $I_{c^*_\varepsilon} - I_{c^*}$ has the same signature for all $\varepsilon$ is that $\delta I$ be equal to zero. If $\delta I = 0$, $\delta^2 I > 0$, then $I_{c^*}$ is minimum, if $\delta I = 0$, $\delta^2 I < 0$, then $I_{c^*}$ is maximum.

The sufficient condition that $\delta I = 0$ is that the expression under integral (2.7) is equal to zero, but it is not a tensor equation. It will be a tensor for some special case of $\delta r$, namely if

$$dy^{Aa} = v^{Aa}dt, \quad A = \overline{0,k}, \qquad dp_{\alpha a} = h_{\alpha a}dt, \quad \alpha = \overline{1,k}.$$

In this case the sufficient condition for $\delta I = 0$ is

$$[dy^{0a}\partial_{0a} + dy^{1a}\partial_{1a} + \cdots + dy^{ka}\partial_{ka} + dp_{1a}\partial^{1a} + \cdots + dp_{ka}\partial^{ka}]F = 0,$$

which can be written in the form

$$\left[y^{1a}\partial_{0a} + y^{2a}\partial_{1a} + \cdots + \frac{dy^{ka}}{dt}\partial_{ka} + p_{2a}\partial^{1a} + \cdots + \frac{dp_{ka}}{dt}\partial^{ka}\right] F = 0$$

or

$$\frac{dF}{dt} = 0 \Leftrightarrow \Gamma_k F = 0,$$

where $\Gamma_k$ is defined in [9].

In some books, the notation $v^{Aa} = \delta y^{Aa}$, $A = \overline{0,k}$ is used and it is called the variation of the variable $y^{Aa}$. Sometimes it is written as $\delta x, \delta\dot{x}, \delta\ddot{x}, \ldots$.

For the further examination we shall introduce the notations:

(2.8)
$$I_1'(v) = \binom{k}{k} v^{0a}\partial_{ka}$$

$$I_2'(v) = \binom{k-1}{k-1} v^{0a}\partial_{(k-1)a} + \binom{k}{k-1} v^{1a}\partial_{ka}, \ldots,$$

$$I_k'(v) = \binom{1}{1} v^{0a}\partial_{1a} + \binom{2}{1} v^{1a}\partial_{2a} + \cdots + \binom{k}{1} v^{(k-1)a}\partial_{ka},$$

$$I_2''(h) = \binom{k-1}{k-1} h_{1a}\partial^{ka}$$

$$I_3''(h) = \binom{k-2}{k-2} h_{1a}\partial^{(k-1)a} + \binom{k-1}{k-2} h_{2a}\partial^{ka}, \ldots,$$

$$I_k''(h) = \binom{1}{1} h_{1a}\partial^{2a} + \binom{2}{1} h_{2a}\partial^{3a} + \cdots + \binom{k-1}{1} h_{(k-1)a}\partial^{ka}.$$

If the space $(GLH)^{(nk)}$ reduces to the generalized Lagrange space $(GL)^{(nk)}$ from (2.8) we can see that $I_1'(v), I_2'(v), \ldots, I_k'(v)$ are equal to $I_V^1, I_V^2, \ldots, I_V^k$ used by R. Miron in [13, 14] if we substitute $v^{0i}$ by $V^i$ and $\frac{y^{Ai}}{A!}$ by $y^{Ai}$.

Let us introduce the notations:

(2.9)
$$\bar{E}^0_a = \partial_{0a} - d^1_t\partial_{1a} + d^2_t\partial_{2a} - \cdots + (-1)^k d^k_t \partial_{ka},$$

$$\overline{\overline{E}}^a_1 = \partial^{1a} - d^1_t\partial^{2a} + d^2_t\partial^{3a} - \cdots + (-1)^{k-1} d^{k-1}_t \partial^{ka}.$$

Using the above notations we can state the important identity given by

**Theorem 2.1.** *The following relation is valid:*

(2.10)
$$v^{0a}\partial_{0a} + v^{1a}\partial_{1a} + \cdots + v^{ka}\partial_{ka} + h_{1a}\partial^{1a} + \cdots + h_{ka}\partial^{ka} =$$

$$v^{0a}\bar{E}^0_a + h_{1a}\overline{\overline{E}}^a_1 + d^1_t(I'_k(v) + I''_k(h)) - d^2_t(I'_{k-1}(v) + I''_{k-1}(h)) +$$

$$\cdots + (-1)^{k-2} d^{k-1}_t (I'_2(v) + I''_2(h)) + (-1)^k d^k_t I'_1(v).$$

**Remark.** In $(GL)^{(nk)}$ (2.10) is shorter, because in this space $h_{1a}\partial^{1a} + \cdots + h_{ka}\partial^{ka} = 0, \overline{\overline{E}}^a_1 = 0, I''_k(h) = 0, I''_{k-1}(h) = 0, \ldots, I''_2(h) = 0$.

*Proof.* For the general case the proof is based on the following property of binomial coefficients:

$$\sum_{n=a}^{n=b} (-1)^n \binom{n}{a}\binom{b}{n} = 0 \qquad a < b,$$

$$a, b \in \{0, 1, 2, \ldots\}.$$

From (2.7) and (2.10) we get

(2.11)
$$\delta I = \int_0^1 (v^{0a}\bar{E}^{0a} + h_{1a}\overline{\overline{E}}^a_1) F dt.$$

□

**Theorem 2.2.** *The sufficient condition that $I_{c^*}$ be the extremal value of $I_{c^*_\varepsilon}$ in $(GLH)^{(nk)}$ is the following equation:*

(2.12)
$$(v^{0a}\bar{E}^0_a + h_{1a}\overline{\overline{E}}^1_1) F = 0.$$

For the special case we have

**Theorem 2.3.** *For $v^{0a} = y^{1a}$ and $h_{1a} = p_{2a}$ in $(GLH)^{(nk)}$ we have*

$$y^{1a}\bar{E}^0_a + p_{2a}\overline{\overline{E}}^a_1 = y^{1a'}\bar{E}^0_{a'} + p_{2a'}\overline{\overline{E}}^{a'}_1,$$

*i.e. the left-hand side of* (2.12) *is a scalar field.*

Moreover, $\bar{E}^0_a$ and $\overline{\overline{E}}^a_1$ will be given in the next section.

## 3. Craig-Synge vectors and covectors

In 1935, Craig and Synge defined covector fields $\overset{(i)}{E}_a$, $i=\overline{0,k}$, in [4] and [19] which were connected with the higher order Finsler spaces. Similar covector fields are given in R. Miron's books [13], [14], ... and they are connected with Lagrange spaces of order $k$. Here, they will be examined in generalized Lagrange-Hamilton spaces $(GLH)^{(nk)}$. In these spaces we obtain two kinds of families: one of vector fields and the other "covector" fields.

Let us consider the curve $c^* : t \in [0,1] \to c^*(t) \in (GLH)^{(nk)}$ and the differentiable fundamental function $F = F(y^0, y^1, \dots, y^k, p_1, \dots, p_k)$. Now we have

**Definition 3.1.** *The Craig-Synge "covectors" in $(GLH)^{(nk)}$ along the curve $c^*(t)$ are defined by*

(3.1)

$$\begin{aligned}
\bar{E}^0_a(F) &= \left[\tbinom{0}{0}\partial_{0a} - \tbinom{1}{0}d^1_t\partial_{1a} + \tbinom{2}{0}d^2_t\partial_{2a} - \cdots + (-1)^k\tbinom{k}{0}d^k_t\partial_{ka}\right](F),\\
\bar{E}^1_a(F) &= \left[-\tbinom{1}{1}\partial_{1a} + \tbinom{2}{1}d^1_t\partial_{2a} - \cdots + (-1)^k\tbinom{k}{1}d^{k-1}_t\partial_{ka}\right](F),\\
\bar{E}^2_a(F) &= \left[\tbinom{2}{2}\partial_{2a} - \cdots + (-1)^k\tbinom{k}{2}d^{k-2}_t\partial_{ka}\right](F), \dots,\\
\bar{E}^k_a(F) &= (-1)^k\tbinom{k}{k}\partial_{ka}(F).
\end{aligned}$$

Formally, $\bar{E}^A_a$, $A=\overline{0,k}$ are the same as the corresponding covectors in the Lagrange spaces of order $k$ (see (8.4.1) in [13], only here $y^{Aa} = d^A_t y^{0a}$). The main difference is the fact, that in $(GLH)^{(nk)}$ $\partial_{Aa}$, $A=\overline{0,k}$ have different transformation law (see (1.4)). From this it follows

**Theorem 3.1.** *In $(GLH)^{(nk)}$ $\bar{E}^0_a$ defined by* (3.1) *is not covector.*

*Proof.* Let us restrict the proof for $k=1$. Then, using (1.4) we get

$$\begin{aligned}
\text{(3.2)}\qquad \bar{E}^0_a &= \partial_{0a} - d^1_t\partial_{1a}\\
&= (\partial_{0a}y^{0a'})\partial_{0a'} + (\partial_{0a}y^{1a'})\partial_{1a'} + (\partial_{0a}p_{1a'})\partial^{1a'} -\\
&\quad -d^1_t[\partial_{1a}y^{1a'})\partial_{1a'}].
\end{aligned}$$

We have

$$y^{1a'} = B^{a'}_a y^{1a}, \quad B^{a'}_a = \partial_{0a}y^{0a'}, \quad \partial_{1a}y^{1a'} = B^{a'}_a,$$

$$(\partial_{0a}y^{1a'})\partial_{1a'} = B^{a'}_{ab}y^{1b}\partial_{1a'}$$

$$d^1_t[(\partial_{1a}y^{1a'})\partial_{1a'}] = (B^{a'}_{ab}y^{1b})\partial_{1a'} + B^{a'}_a d^1_t\partial_{1a'}.$$

Substituting the last two equations into (3.2) we get

$$\bar{E}^0_a = B^{a'}_a(\partial_{0a'} - d^1_t\partial_{1a'}) + (\partial_{0a}p_{1a'})\partial^{1a'}$$

$$= B_a^{a'} \bar{E}^0_{a'} + (\partial_{0a} p_{1a'}) \partial^{1a'}.$$

The above equation proves Theorem 3.1. □

If $(GLH)^{(nk)}$ reduces to $(GL)^{(nk)}$, then in (1.4) terms of the form $\partial_{Aa} p_{\alpha a'}$ $\alpha \geq A$ do not appear, and we obtain the known result: (see [13])

**Theorem 3.2.** *$\bar{E}^0_a$, defined by (3.2) in generalized Lagrange space $(GL)^{(nk)}$, is a covector.*

**Proposition 3.1.** *If $\phi = \phi(y^0, y^1, \ldots, y^k, p_1, p_2, \ldots, p_k)$ is a differentiable function in $(GLH)^{(nk)}$, such that $\partial_{ka}\phi = 0$, $\partial^{ka}\phi = 0$, then*

(3.3)
$$\partial_{0a} d^1_t \phi = (d^1_t \partial_{0a})\phi,$$
$$\partial_{1a} d^1_t \phi = (\partial_{0a} + d^1_t \partial_{1a})\phi,$$
$$\partial_{2a} d^1_t \phi = (\partial_{1a} + d^1_t \partial_{2a})\phi, \ldots,$$
$$\partial_{(k-1)a} d^1_t \phi = (\partial_{(k-2)a} + d^1_t \partial_{(k-1)a})\phi,$$
$$\partial_{ka} d^1_t \phi = \partial_{(k-1)a}\phi,$$

(3.4)
$$\partial^{1a}(d^1_t \phi) = (d^1_t \partial^{1a})\phi,$$
$$\partial^{2a}(d^1_t \phi) = (\partial^{1a} + d^1_t \partial^{2a})\phi, \ldots,$$
$$\partial^{(k-1)a}(d^1_t \phi) = (\partial^{(k-2)a} + d^1_t \partial^{(k-1)a})\phi,$$
$$\partial^{ka}(d^1_t \phi) = \partial^{(k-1)a}\phi.$$

*Proof.* Using the assumptions $\partial_{ka}\phi = 0$, $\partial^{ka}\phi = 0$, we have

(3.5)
$$\begin{aligned} d^1_t \phi &= [(y^{1b}\partial_{0b} + y^{2b}\partial_{1b} + \cdots + y^{kb}\partial_{(k-1)b}) + \\ &\quad (p_{2b}\partial^{1b} + p_{3b}\partial^{2b} + \cdots + p_{kb}\partial^{(k-1)b})]\phi, \\ \partial_{0a} d^1_t &= [(y^{1b}\partial_{0a}\partial_{0b} + y^{2b}\partial_{0a}\partial_{1b} + \cdots + y^{kb}\partial_{0a}\partial_{(k-1)b}) + \\ &\quad (p_{2b}\partial_{0a}\partial^{1b} + p_{3b}\partial_{0a}\partial^{2b} + \cdots + p_{kb}\partial_{0a}\partial^{(k-1)b}]\phi. \end{aligned}$$

From the above two equations it follows $\partial_{0a} d^1_t \phi = d^1_t \partial_{0a}\phi$, which is the first equation of (3.3). From (3.5) it follows

$$\begin{aligned} \partial_{1a} d^1_t \phi &= [\partial_{0a} + (y^{1b}\partial_{1a}\partial_{0b} + y^{2b}\partial_{1a}\partial_{1b} + \cdots + y^{kb}\partial_{1a}\partial_{(k-1)b}) + \\ &\quad (p_{2b}\partial_{1a}\partial^{2b} + p_{3b}\partial_{1a}\partial^{2b} + \cdots + p_{kb}\partial_{1a}\partial^{(k-1)b})]\phi. \end{aligned}$$

From the above equation it follows

$$\partial_{1a} d_t^1 \phi = (\partial_{0a} + d_t^1 \partial_{1a})\phi,$$

which is the second equation of (3.3). As $\partial_{ka}\phi = 0$, from (3.5) it follows

$$\partial_{ka}(d_t^1 \phi) = (\partial_{ka} y^{kb})\partial_{(k-1)b}\phi = \partial_{(k-1)a}\phi,$$

which is the last equation of (3.3). (3.4) can be proved using the same method. □

**Proposition 3.2.** *If $\phi = \phi(y^0, y^1, \dots, y^k, p_1, \dots, p_k)$ is a differentiable function in $(GLH)^{(nk)}$, such that $\partial_{ka}\phi = 0$, $\partial^{ka}\phi = 0$, then*

$$\begin{aligned}
\bar{E}_a^0(d_t^1 \phi) &= 0 \\
\bar{E}_a^1(d_t^1 \phi) &= -\bar{E}_a^0(\phi) \\
\bar{E}_a^2(d_t^1 \phi) &= -\bar{E}_a^1(\phi), \dots, \\
\bar{E}_a^k(d_t^1 \phi) &= -\bar{E}_a^{(k-1)}\phi.
\end{aligned} \tag{3.6}$$

*The above equations are the extensions of the results of Caratheodory [3].*

*Proof.* Using (3.3) and (3.1) we obtain:

$$\begin{aligned}
\bar{E}_a^0(d_t^1\phi) &= (\partial_{0a} - d_t^1\partial_{1a} + d_t^2\partial_{2a} + \dots + (-1)^k d_t^k \partial_{ka})(d_t^1\phi) \\
&= [d_t^1\partial_{0a} - d_t^1(\partial_{0a} + d_t^1\partial_{1a}) + d_t^2(\partial_{1a} + d_t^1\partial_{2a}) \\
&\quad -d_t^3(\partial_{2a} + d_t^1\partial_{3a}) + \dots + (-1)^{k-1} d_t^{k-1}(\partial_{(k-2)a} + d_t^1\partial_{(k-1)a}) \\
&\quad +(-1)^k d_t^k \partial_{(k-1)a}]\phi.
\end{aligned}$$

From the above it follows

$$\bar{E}_a^0(d_t^1\phi) = 0.$$

Using the well known relation: $\binom{n}{k} + \binom{n}{k-1} = \binom{n+1}{k}$ (3.1) and (3.3) we have:

$\bar{E}_a^1(d_t^1\phi)$

$$\begin{aligned}
&= [-\binom{1}{1}\partial_{1a} + \binom{2}{1} d_t^1\partial_{2a} - \binom{3}{1} d_t^2\partial_{3a} + \dots + (-1)^k \binom{k}{1} d_t^{k-1}\partial_{ka}](d_t^1\phi) \\
&= [-\binom{1}{1}(\partial_{0a} + d_t^1\partial_{1a}) + \binom{2}{1} d_t^1(\partial_{1a} + d_t^1\partial_{2a}) - \binom{3}{1} d_t^2(\partial_{2a} + d_t^1\partial_{3a}) + \cdots \\
&\quad +(-1)^{k-1}\binom{k-1}{1} d_t^{k-2}(\partial_{(k-2)a} + d_t^1\partial_{(k-1)a}) + (-1)^k \binom{k}{1} d_t^{(k-1)}\partial_{(k-1)a}]\phi
\end{aligned}$$

$$= [-\binom{0}{0}\partial_{0a} + [\binom{2}{1} - \binom{1}{1}]d_t^1\partial_{1a} - [\binom{3}{1} - \binom{2}{1}]d_t^2\partial_{2a} + [\binom{4}{1} - \binom{3}{1}]d_t^3\partial_{3a} - \cdots$$

$$+(-1)^k[\binom{k}{1} - \binom{k-1}{1}]d_t^{k-1}\partial_{(k-1)a} + (-1)^{k+1}\binom{k}{0}d_t^k\partial_{ka}]\phi.$$

The last term is equal to zero, because $\partial_{ka}\phi = 0$, so we obtain

$$\begin{aligned}\bar{E}_a^1(d_t^1\phi) &= -[\binom{0}{0}\partial_{0a} - \binom{1}{0}d_t^1\partial_{1a} + \binom{2}{0}d_t^2\partial_{2a} + \binom{3}{0}d_t^3\partial_{3a} - \cdots + \\ &\quad (-1)^{k-1}\binom{k-1}{0}d_t^{k-1}\partial_{(k-1)a} + (-1)^k\binom{k}{0}d_t^k\partial_{ka}]\phi,\end{aligned}$$

i.e.

$$\bar{E}_a^1(d_t^1\phi) = -\bar{E}_a^0\phi.$$

The other relations from (3.6) can be proved in the same way. □

In $(GLH)^{(nk)}$ we can define vector fields by

**Definition 3.2.** *If $F(y^0, y^1, \ldots, y^k, p_1, \ldots, p_k)$ is a differentiable function in $(GLH)^{(nk)}$, then along the curve $c^*(t)$ the Craig-Synge vector fields $\overline{\overline{E}}{}_\alpha^a$, $\alpha = \overline{1,k}$, are defined by*

(3.7)

$$\begin{aligned}\overline{\overline{E}}{}_1^a(F) &= \left(\tbinom{0}{0}\partial^{1a} - \tbinom{1}{0}d_t^1\partial^{2a} + \tbinom{2}{0}d_t^2\partial^{3a} - \cdots + (-1)^{k-1}\tbinom{k-1}{0}d_t^{k-1}\partial^{ka}\right)F,\\ \overline{\overline{E}}{}_2^a(F) &= \left(-\tbinom{1}{1}\partial^{2a} + \tbinom{2}{1}d_t^1\partial^{3a} - \cdots + (-1)^{k-1}\tbinom{k-1}{1}d_t^{k-2}\partial^{ka}\right)F,\\ \overline{\overline{E}}{}_3^a(F) &= \left(\tbinom{2}{2}\partial^{3a} - \cdots + (-1)^{k-1}\tbinom{k-1}{2}d_t^{k-3}\partial^{ka}\right)F, \ldots,\\ \overline{\overline{E}}{}_k^a(F) &= (-1)^{k-1}\tbinom{k-1}{k-1}\partial^{ka}F.\end{aligned}$$

**Proposition 3.3.** *If $\phi(y^0, y^1, \ldots, y^k, p_1, \ldots, p_k)$ is a differentiable function in $(GLH)^{(nk)}$, such that $\partial_{ka}\phi = 0$, $\partial^{ka}\phi = 0$, then*

(3.8)
$$\begin{aligned}&\overline{\overline{E}}{}_1^a(d_t^1\phi) = 0\\ &\overline{\overline{E}}{}_2^a(d_t^1\phi) = -\overline{\overline{E}}{}_1^a(\phi)\\ &\overline{\overline{E}}{}_3^a(d_t^1\phi) = -\overline{\overline{E}}{}_2^a(\phi)\\ &\overline{\overline{E}}{}_k^a(d_t^1\phi) = -\overline{\overline{E}}{}_{k-1}^a(\phi).\end{aligned}$$

*Proof.* Using (3.4), (3.7) we have

$$\begin{aligned}&\overline{\overline{E}}{}_1^a(d_t^1\phi)\\ &= (\partial^{1a} - d_t^1\partial^{2a} + d_t^2\partial^{3a} - \cdots + (-1)^{k-1}d_t^{k-1}\partial^{ka})(d_t^1\phi)\end{aligned}$$

$$= (d_t^1\partial^{1a} - d_t^1(\partial^{1a} + d_t^1\partial^{2a}) + d_t^2(\partial^{2a} - d_t^1\partial^{3a}) + \cdots$$
$$\cdots + (-1)^{k-1} d_t^{k-1}\partial^{(k-1)a})\phi = 0.$$

$$\overline{\overline{E}}{}_2^a(d_t^1\phi)$$
$$= (-\binom{1}{1}\partial^{2a} + \binom{2}{1}d_t^1\partial^{3a} - \binom{3}{1}d_t^2\partial^{4a} + \cdots + (-1)^{k-1}\binom{k-3}{1}d_t^{k-2}\partial^{ka})(d_t^1\phi)$$
$$= [-\binom{1}{1}(\partial^{1a} + d_t^1(\partial^{2a}) + \binom{2}{1}d_t^1\partial^{2a} + d_t^1\partial^{3a}) - \binom{3}{1}d_t^2(\partial^{3a} - d_t^1\partial^{4a}) + \cdots]\phi$$
$$= [-\binom{0}{0}\partial^{1a} + [\binom{2}{1} - \binom{1}{1}]d_t^1\partial^{2a} - [\binom{3}{1} - \binom{2}{1}]d_t^2\partial^{3a} + \cdots]\phi = -\overline{\overline{E}}{}_1^a(\phi), \ldots,$$

$$\overline{\overline{E}}{}_k^a(d_t^1\phi) = [(-1)^{k-1}\binom{k-1}{k-1}\partial^{(k-1)a}$$
$$= -[(-1)^{k-2}\binom{k-2}{k-2}\partial^{(k-1)a} + (-1)^{(k-1)}\binom{k-1}{k-2}d_t^1\partial_a^k\phi.$$

The last term, which was added, is equal to zero because $\partial^{ka}\phi = 0$. So we obtain

$$\overline{\overline{E}}{}_k^a(d_t^1\phi) = -\overline{\overline{E}}{}_{k-1}^a(\phi).$$

□

**Consequence**:

**Theorem 3.3.** *In $(GLH)^{(nk)}$ the integrals of actions*

$$I_{c^*} = \int_0^1 F(y^0, y^1, \ldots, y^k, p_1, \ldots, p_k)$$

$$I'_{c^*} = \int_0^1 [F(y^0, y^1, \ldots, y^k, p_1, \ldots, p_k) + d_t^1\phi(y^0, y^1, \ldots, y^{k-1}, p_1, \ldots, p_{k-1})]dt$$

*have the same extremal curves for any differentiable fundamental function $F$ and any differentiable function $\phi$, for which*

$$\partial_{ka}\phi = 0, \quad \partial^{ka}\phi = 0.$$

*Proof.* The extremal curves of $I_{c^*}$ are the solution of

$$(v^{0a}\bar{E}_a^0 + h_{1a}\overline{\overline{E}}{}_1^a)F = 0$$

and those of $I_{c^*}$ satisfy

$$(v^{0a}\bar{E}^0_a + h_{1a}\overline{\overline{E}}_1{}^a)(F + d^1_t\phi) = 0.$$

As $\bar{E}^0_a(d^1_t\phi) = 0$, $\overline{\overline{E}}_1{}^a(d^1_t\phi) = 0$, (see (3.6) and (3.8)), the extremal curves for both integrals are the solution of the same differential equation. □

**Proposition 3.4.** *If $\phi = \phi(t)$ and $F = F(y^0, y^1, \ldots, y^k, p_1, \ldots, p_k)$ are differentiable functions in $(GLH)^{(nk)}$, then the following relations are valid*

$$\begin{aligned}\bar{E}^0_a(\phi F) \quad &= \quad \phi\bar{E}^0_a(F) + (d^1_t\phi)\bar{E}^1_a(F) + \cdots + \\ &\quad (d^{k-1}_t\phi)E^{(k-1)}_a(F) + (d^k_t\phi)\bar{E}^k_a.\end{aligned} \tag{3.9}$$

*Proof.* From

$$\begin{aligned}\bar{E}^0_a(\phi F) \quad &= \quad [\binom{0}{0}\partial_{0a} - \binom{1}{0}d^1_t\partial_{1a} + \binom{2}{0}d^2_t\partial_{2a} - \binom{3}{0}d^3_t\partial_{3a} + \\ &\quad \binom{4}{0}d^4_t\partial_{4a} + \cdots + (-1)^k\binom{k}{0}d^k_t\partial_{ka}](\phi F).\end{aligned}$$

and $\phi = \phi(t)$ we have

$$\begin{aligned}\partial_{0a}(\phi F) \quad &= \quad \binom{0}{0}\phi\partial_{0a}F, \\ -d^1_t(\partial_{1a}(\phi F)) \quad &= \quad -d^1_t(\phi\partial_{1a}F) = -[\binom{1}{1}(d^1_t\phi)\partial_{1a} + \binom{1}{0}\phi d^1_t\partial_{1a}]F, \\ d^2_t(\phi\partial_{2a}F) \quad &= \quad [\binom{2}{2}(d^2_t\phi)\partial_{2a} + \binom{2}{1}(d^1_t\phi)d^1_t\partial_{2a} + \binom{2}{0}\phi d^2_t\partial_{2a}]F, \\ -d^3_t(\phi\partial_{3a}F) \quad &= \quad -[\binom{3}{0}(d^3_t\phi)\partial_{3a} + \binom{3}{2}(d^2_t\phi)d^1_t\partial_{3a} \\ &\quad +\binom{3}{1}(d^1_t\phi)d^2_t\partial_{3a} + \binom{3}{0}\phi d^3_t\partial_{3a}]F, \ldots, \\ (-1)^k d^k_t(\phi\partial_{ka}F) \quad &= \quad (-1)^k[\binom{k}{k}(d^k_t\phi)\partial_{ka} \\ &\quad +\binom{k}{k-1}d^{k-1}_t\phi d^1_t\partial_{ka} + \cdots + \binom{k}{0}\phi d^k_t\partial_{ka}]F.\end{aligned}$$

The addition of former equations results (in the first line are the last terms, and so on) in:

$$\begin{aligned}\bar{E}^0_a(\phi F) = \phi[\tbinom{0}{0}\partial_{0a} - \tbinom{1}{0}d^1_t\partial_{1a} + \tbinom{2}{0}d^2_t\partial_{2a} - \tbinom{3}{0}d^3_t\partial_{3a} + \cdots + (-1)^k\tbinom{k}{0}d^k_t\partial_{ka}]F+ \\ (d^1_t\phi)[-\tbinom{1}{1}\partial_{1a} + \tbinom{2}{1}d^1_t\partial_{2a} - \tbinom{3}{1}d^2_t\partial_{3a} + (-1)^k\tbinom{k}{1}d^{k-1}_t\partial_{ka}]F+ \\ (d^2_t\phi)[\tbinom{2}{2}\partial_{2a} - \tbinom{3}{2}d^1_t\partial_{3a} + \cdots + (-1)^k\tbinom{k}{2}d^{k-2}_t\partial_{ka}]F+ \\ (d^3_t\phi)[-\tbinom{3}{3}\partial_{3a} + \cdots + (-1)^k\tbinom{k}{3}d^{k-3}_t\partial_{ka}]F+ \\ \cdots + (d^k_t\phi)(-1)^k\tbinom{k}{k}\partial_{ka}F.\end{aligned}$$

The comparison of the above equation with (3.1) gives (3.9). □

As a consequence of the previous proposition we have:

**Proposition 3.5.** *In $(GLH)^{(nk)}$, the following relations are valid*

$$
\begin{aligned}
&\bar{E}^0_a(F) = \bar{E}^0_a(F),\\
&\bar{E}^0_a(tF) = t\bar{E}^0_a(F) + \bar{E}^1_a(F),\\
&\bar{E}^0_a(t^2F) = t^2E^0_a(F) + 2t\bar{E}^1_a(F) + 2!\bar{E}^2_a(F),\\
&\bar{E}^0_a(t^kF) = t^kE^0_a(F) + kt^{k-1}\bar{E}^1_aF + \cdots + k!\bar{E}^k_aF.
\end{aligned}
\tag{3.10}
$$

**Theorem 3.4.** *If $(GLH)^{(nk)}$ is reduced to $(GL)^{(nk)}$, then $\bar{E}^0_a, \bar{E}^1_a, \dots, \bar{E}^k_a$ are covectors.*

*Proof.* It is known that $\bar{E}^0_a$ in $(GL)^{(nk)}$ is covector (Theorem 3.2). From the second equation of (3.10) we get

$$\bar{E}^1_a(F) = \bar{E}^0_a(tF) - t\bar{E}^0_a(F) = B^{a'}_a(\bar{E}^0_{a'}(tF) - t\bar{E}^0_a(F)) = B^{a'}_a\bar{E}^1_{a'}(F), \dots .$$

From the above it follows that $\bar{E}^1_a$ is a covector. For $\phi = t^2$ we get

$$\bar{E}^0_a(t^2F) = t^2\bar{E}^0_a(F) + 2t\bar{E}^1_a(F) + 2\bar{E}^2_a(F),$$

from which we conclude that $\bar{E}^2_a$ is a covector, and so on. □

**Theorem 3.5.** *If $(GLH)^{(nk)}$ is reduced to $(SLH)^{(nk)}$ then $\bar{E}^0_a, \bar{E}^1_a, \dots, \bar{E}^k_a$ are covectors.*

*Proof.* From (1.7) we have:

$$\partial_{Aa} = B^{a'}_a\partial_{Aa}, \qquad A = \overline{0,k}, d^A_t B^{a'}_a = 0, A = \overline{0,k}.$$

□

**Proposition 3.6.** *If $\phi = \phi(t)$ and $F = F(y^0, y^1, \dots, y^k, p_1, \dots, p_k)$ are differentiable functions in $(GLH)^{(nk)}$, then the following relations are valid*

$$\overline{\overline{E}}_1{}^a(\phi F) = \phi\overline{\overline{E}}_1{}^a(F) + (d^1_t\phi)\overline{\overline{E}}_2{}^a(F) + \cdots + (d^{(k-1)}_t\phi)\overline{\overline{E}}_a{}^k(F). \tag{3.11}$$

*Proof.* As $\phi = \phi(t)$ we have $\partial^{\alpha a}(\phi F) = \phi\partial^{\alpha a}F$, $\alpha = \overline{1,k}$. We get

$$\overline{\overline{E}}_1{}^a(\phi F) = [\binom{0}{0}\partial^{1a} - \binom{1}{0}d^1_t\partial^{2a} + \binom{2}{0}d^2_t\partial^{3a} - \cdots + (-1)^{k-1}\binom{k-1}{0}d^{k-1}_t\partial^{ka}](\phi F).$$

If we add all the following equations

$$\binom{0}{0}\partial^{1a}(\phi F) \;=\; \binom{0}{0}\phi\partial^{1a}F,$$

$$
\begin{aligned}
-\binom{1}{0}d_t^1\partial^{2a}(\phi F) &= -\binom{1}{0}[\binom{1}{1}(d_t^1\phi)\partial^{2a}+\binom{1}{0}\phi d_t^1\partial^{2a}]F,\\
+\binom{2}{0}d_t^2\partial^{3a}(\phi F) &= \binom{2}{0}[\binom{2}{2}(d_t^2\phi)\partial^{3a}+\binom{2}{1}d_t^1\phi d_t^1\partial^{3a}\\
&\quad+\binom{2}{0}\phi d_t^2\partial^{3a}]F,\ldots,\\
(-1)^{k-1}\binom{k-1}{0}d_t^{k-1}\partial^{ka}(\phi F) &= (-1)^{k-1}\binom{k-1}{0}[\binom{k-1}{k-1}(d_t^{k-1}\phi)\partial^{ka}\\
&\quad+\binom{k-1}{k-2}(d_t^{k-2}\phi)d_t^1\partial^{ka}+\cdots+\binom{k-1}{0}\phi d_t^{k-1}\partial^{ka}]F,
\end{aligned}
$$

we obtain

$$
\begin{aligned}
\overline{\overline{E}}_1^{\,a}(\phi F) &= \phi[\binom{0}{0}\partial^{1a}-\binom{1}{0}d_t^1\partial^{2a}+\binom{2}{0}d_t^2\partial^{3a}-\cdots\\
&\quad\cdots+(-1)^{k-1}\binom{k-1}{0}d_t^{k-1}\partial^{ka}]F+\\
&\quad(d_t^1\phi)[-\binom{1}{1}\partial^{2a}+\binom{2}{1}d_t^1\partial^{3a}+\cdots\\
&\quad\cdots+(-1)^{k-1}\binom{k-1}{1}d_t^{k-1}\partial^{ka}]F+\cdots+\\
&\quad(d_t^{k-1}\phi)(-1)^{k-1}\binom{k-1}{k-1}\partial^{ka}F.
\end{aligned}
$$

□

The comparison of the above equation with (3.7) gives (3.11).

**Theorem 3.6.** *In* $(GLH)^{(nk)}\overline{\overline{E}}_1^a,\ldots,\overline{\overline{E}}_k^a$ *are vector fields.*

*Proof.* $\overline{\overline{E}}_1^a$ is a vector field because $\partial^{1a},\partial^{2a},\ldots,\partial^{ka}$ in $(GLH)^{(nk)}$ have the similar transformation law as $\partial_{0a},\ldots,\partial_{ka}$ in $(GL)^{(nk)}$, where $(\partial_{Aa}p_{\alpha a'})=0$, $A<\alpha$. □

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The generalised Euler-Lagrange equations and constraint equa-tions are derived directly from the variation of the action on the space of dependent variables. This set of Lagrangian equations gives rise to a crucial property of integrable systems known as the multidimensional consistency.Â  The variational principle on the phase space will be considered resulting in generalised Hamiltonâ€™s equations and commuting Poisson bracket which consequently gives us the commuting Hamiltonian ows. In section 5, NÂ¨oether charges are directly derived from the variation of action functional of the Lagrangian 1-form. We nd that all NÂ¨oether charges are nothing but all Hamiltonians (invariants) in the system. Introduced by the Italian-French mathematician and astronomer Joseph-Louis Lagrange in 1788, Lagrangian mechanics is a formulation of classical mechanics and is founded on the stationary action principle. Given a system of point masses and a pair, and. of time instants, Lagrangian mechanics postulates that the system's trajectory (represented by a curve in the configuration space) must be a stationary point of the action functional. Cite this chapter as: (2002) Generalized Hamilton spaces of order 2. In: The Geometry of Hamilton and Lagrange Spaces. Fundamental Theories of Physics, vol 118. Springer, Dordrecht. https://doi.org/10.1007/0-306-47135-3_11. Lagrange (or simply Euler) equation. The essential feature is that this equation is equivalent to the original variational problem (when boundary conditions are included). Note also that in this formalism it is essential that the variations in the. path, represented by the function h. ( x. ) , are unconstrained except at the endpoints. Only in this case do we obtain the local differential equation of Eq. (20.8). First we consider certain situations in which we can integrate the Euler-Lagrange equation once trivially. The book begins by applying Lagrangeâ€™s equations to a number of mechanical systems. It introduces the concepts of generalized coordinates and generalized momentum. Following this, the book turns to the calculus of variations to derive the Eulerâ€“Lagrange equations. It introduces Hamiltonâ€™s principle and uses this throughout the book to derive further results. The Hamiltonian, Hamiltonâ€™s equations, canonical transformations, Poisson brackets and Hamiltonâ€“Jacobi theory are considered next. The book concludes by discussing continuous Lagrangians and Hamiltonians and how they are related to eld the