COMPOSITION OPERATORS ON ANALYTIC WEIGHTED HILBERT SPACES

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Abstract. We consider composition operators in the analytic weighted Hilbert space. Various criteria on boundedness, compactness and Hilbert-Schmidt class membership are established.

1. Composition operators on the Hardy space

In this expository paper, we consider composition operators acting on the weighted Hilbert spaces of analytic functions on the unit disc. This paper is based on [5, 6]. A comprehensive study of composition operators in function spaces and their spectral behavior could be found in [3, 11, 15]. See also [4, 8, 9, 13, 14, 16] for a treatment of some of the questions addressed in this paper.

Let φ be an analytic map of the unit disk $\mathbb{D}$ of the complex plane into itself. We define the composition operator C_φ by

$$C_\varphi(f) = f \circ \varphi$$

for f analytic in $\mathbb{D}$.

We recall that the Hardy space H^2 consists of those analytic functions $f : \mathbb{D} \to \mathbb{C}$ for which the norm

$$\|f\|_2 = \sup_{0<r<1} \Big(\frac{1}{2\pi}\int_{\mathbb{T}} |f(r\zeta)|^2 |d\zeta|\Big)^{1/2}$$

is finite. Every function $f \in H^2$ has non–tangential limits almost everywhere on $\mathbb{T}$. We denote by $f(\zeta)$ the non–tangential limit of f at $\zeta \in \mathbb{T}$ if it exists.

1.1. **Boundedness.** It is a consequence of Littlewoods subordination principle [11, 15] that every composition operator C_φ restricts to a bounded operator on H^2. Furthermore

$$\|C_\varphi(f)\|_2^2 \leq \frac{1+|\varphi(0)|}{1-|\varphi(0)|}\|f\|_2^2.$$

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1.2. **Compactness.** Shapiro gives in [10] a complete characterization of compact composition operators in the Hardy space in term of Nevanlinna counting functions

Let $\varphi \in \mathrm{Hol}(\mathbb{D})$ such that $\varphi(\mathbb{D}) \subset \mathbb{D}$. The Nevanlinna counting function is defined for every $z \in \mathbb{D} \setminus \{\varphi(0)\}$ by

$$N_\varphi(z) = \sum_{\substack{\varphi(w)=z \\ w\in\mathbb{D}}} \log \frac{1}{|w|}.$$

where each pre-image w is counted according to its multiplicity. Note that $N_\varphi(z) = O(-\log|z|)$. Shapiro's characterization is as follow

$$C_\varphi \text{ is compact in } H^2 \iff \lim_{|z|\to 1} \frac{N_\varphi(z)}{\log(1/|z|)} = 0.$$

Given $E \subset \mathbb{T}$, we write $|E|$ for the Lebesgue measure of E. Note also that if C_φ is compact, then the level set of φ

$$E_\varphi(1) = \{\zeta \in \mathbb{T} : \, |\varphi(\zeta)| = 1\}$$

has Lebesgue measure zero. Indeed, suppose that $E_\varphi(1)$ has positive measure, we will show that C_φ is not compact. Since the sequence $(z^n)_n$ converges weakly to zero and

$$\begin{aligned} \|C_\varphi(z^n)\|_2^2 &= \|\varphi^n\|_2^2 \\ &= \frac{1}{2\pi}\int_{\mathbb{T}} |\varphi(\zeta)|^{2n}|d\zeta| \\ &\geq \frac{1}{2\pi}\int_{\{\zeta\in\mathbb{T} \,:\, |\varphi(\zeta)|=1\}} |\varphi(\zeta)|^{2n}|d\zeta| \\ &= \frac{|E_\varphi(1)|}{2\pi} > 0, \end{aligned}$$

$\|C_\varphi(z^n)\|_2$ not converge to zero. Hence C_φ is not compact.

1.3. **Hilbert-Schmidt class.** For $s \in (0,1)$, the level set $E_\varphi(s)$ of φ is given by

$$E_\varphi(s) = \{\zeta \in \mathbb{T}\colon \, |\varphi(\zeta)| \geq s\}.$$

One can completely describe the membership of C_φ in the Hilbert-Schmidt class in terms of the size of the level sets of the inducing map φ. Indeed, C_φ is Hilbert-Schmidt in H^2 if and only if

$$\sum_{n\geq 0} \|\varphi^n\|_2^2 = \frac{1}{2\pi}\int_{\mathbb{T}} \frac{|d\zeta|}{1-|\varphi(\zeta)|^2} < \infty.$$

Let f be a measurable function on $\mathbb{T}$, the associated distribution function m_f is given by

$$m_f(\lambda) = |\{\zeta \in \mathbb{T} : \, |f(\zeta)| > \lambda\}|, \qquad \lambda > 0.$$

It then follows that C_φ is in the Hilbert-Schmidt class of H^2 if and only if

$$\int_{\mathbb{T}} \frac{|d\zeta|}{1-|\varphi(\zeta)|^2} = \int_1^\infty m_{(1-|\varphi|^2)^{-1}}(\lambda)\,d\lambda \asymp \int_0^1 \frac{|E_\varphi(s)|}{(1-s)^2}\,ds < \infty.$$

It was shown by Gallardo–Gonzalez in [9] that there exists a a mapping $\varphi : \mathbb{D} \to \mathbb{D}$ such that C_φ is compact in H^2, and that the level set $E_\varphi(1)$ has Hausdorff measure equal to one. Let $A(\mathbb{D})$ denote the disc algebra. In [6] with El-Fallah, Shabankhah and Youssfi we obtain the following result

Theorem 1.1. *Let E be a closed subset of $\mathbb{T}$ with Lebesgue measure zero. There exists a mapping $\varphi : \mathbb{D} \to \mathbb{D}$, $\varphi \in A(\mathbb{D})$ such that C_φ is a Hilbert-Schmidt operator on H^2 and that $E_\varphi(1) = E$.*

2. Composition operators on weighted Hilbert spaces of analytic functions

Given a positive integrable function $\omega \in C^2[0,1)$, we extend it by $\omega(z) = \omega(|z|)$, $z \in \mathbb{D}$, and call such ω a weight function. We denote by $\mathcal{H}_\omega$ the weighted Hilbert space consisting of analytic functions f on $\mathbb{D}$ such that

$$\|f\|^2_{\mathcal{H}_\omega} = |f(0)|^2 + \int_{\mathbb{D}} |f'(z)|^2 \omega(z) dA(z) \ < \ \infty.$$

where $dA(z) = dxdy/\pi$ stands for the normalized area measure in $\mathbb{D}$. A simple computation shows that $f(z) = \sum_{n=0}^\infty a_n z^n \in \mathcal{H}_\omega$, if and only if

$$\|f\|^2_{\mathcal{H}_\omega} = \sum_{n\geq 0} |a_n|^2 w_n < \infty,$$

where $\omega_0 = 1$ and

$$\omega_n = 2n^2 \int_0^1 r^{2n-1}\omega(r)dr, \qquad n \geq 1.$$

Let $\alpha > -1$, $\omega_\alpha(r) = (1-r^2)^\alpha$ and denote $\mathcal{H}_{\omega_\alpha}$ by $\mathcal{H}_\alpha$. The Hardy space H^2 can be identified with $\mathcal{H}_1$. The Dirichlet space $\mathcal{D}_\alpha$ is precisely $\mathcal{H}_\alpha$ for $0 \leq \alpha < 1$ and $\mathcal{H}_0$ corresponds to classical Dirichlet space $\mathcal{D}$. Finally, the Bergman spaces $\mathcal{A}^2_\alpha(\mathbb{D}) := \mathrm{Hol}(\mathbb{D}) \cap \mathrm{L}^2(\mathbb{D}, (1-|z|^2)^\alpha dA(z))$ can be identified with $\mathcal{H}_{\alpha+2}$. From now we assume that the weight ω satisfies the following properties

($\mathcal{W}_1$) ω is non-increasing,
($\mathcal{W}_2$) $\omega(r)(1-r)^{-(1+\delta)}$ is non-decreasing for some $\delta > 0$,
($\mathcal{W}_3$) $\lim\limits_{r\to 1-} \omega(r) = 0$.

Furthermore, we assume that one of the two properties of convexity is fulfilled

($\mathcal{W}_4^{(\mathrm{I})}$) ω is convex and $\lim\limits_{r\to 1} \omega'(r) = 0$.
($\mathcal{W}_4^{(\mathrm{II})}$) ω is concave.

Such a weight function is called admissible. If ω satisfies conditions $(\mathcal{W}_1)$–$(\mathcal{W}_3)$ and $(\mathcal{W}_4^{(\mathrm{I})})$ (resp. $(\mathcal{W}_4^{(\mathrm{II})})$), we shall say that ω is (I)-admissible (resp. (II)-admissible).

Note that (I)-admissibility corresponds to the case $\mathrm{H}^2 \subsetneq \mathcal{H}_\omega \subset \mathcal{A}^2_\alpha(\mathbb{D})$ for some $\alpha > -1$, whereas (II)-admissibility corresponds to the case $\mathcal{D} \subsetneq \mathcal{H}_\omega \subsetneq \mathrm{H}^2$. The weight $\omega_0 = 1$ is not an admissible weight, so the results of this section do not apply to the Dirichlet space.

The Nevanlinna counting functions play a key role here (see [7] for recent results on this topic). Let $\varphi \in \mathrm{Hol}(\mathbb{D})$ such that $\varphi(\mathbb{D}) \subset \mathbb{D}$. The generalized counting Nevanlinna function associated to ω is defined for every $z \in \mathbb{D} \setminus \{\varphi(0)\}$ by

$$N_{\varphi,\omega}(z) = \sum_{\substack{\varphi(a)=z \\ a\in\mathbb{D}}} \omega(a).$$

Let us point out two crucial properties the generalized counting Nevanlinna function :

• If f is positive mesurable function on $\mathbb{D}$ and φ is a holomorphic self-map on $\mathbb{D}$, then

$$\int_{\mathbb{D}} (f \circ \varphi)(z)|\varphi'(z)|^2 w(z) dA(z) = \int_{\mathbb{D}} f(z) N_{\varphi,w}(z) dA(z). \tag{1}$$

• If ω is an admissible weight and φ is a holomorphic self-map of $\mathbb{D}$. Then the generalized Nevanlinna counting function satisfies the sub-mean value property, that is for every $r > 0$ and for every $z \in \mathbb{D}$ such that $D(z, r) \subset \mathbb{D} \setminus D(0, 1/2)$

$$N_{\varphi,\alpha}(z) \leq \frac{2}{r^2} \int_{|\zeta - z|<r} N_{\varphi,\alpha}(\zeta) dA(\zeta). \tag{2}$$

If φ is a holomorphic map on the unit disk $\mathbb{D}$ into itself, it is an easy consequence of Littlewood's subordination principle that the composition operator with φ, induces a bounded operator C_φ on $\mathcal{H}_\omega$ for (I)–admissible weight ω. In [5] with Lefèvre, see also [1], we obtain the following results. For the case of (II)–admissible weight we have

Theorem 2.1. *Let ω be a* (II)*–admissible weight and $\varphi \in \mathcal{H}_\omega$. Then C_φ is bounded on $\mathcal{H}_\omega$ if and only if*

$$\sup_{|z|<1} \frac{N_{\varphi,\omega}(z)}{\omega(z)} < \infty \tag{3}$$

The characterization of compactness in the weighted Bergman spaces $\mathcal{H}_{\alpha+2}$ is given by

$$C_\varphi \text{ is compact in } \mathcal{H}_{\alpha+2} \iff \lim_{|z|\to 1-} \frac{1 - |\varphi(z)|^2}{1 - |z|^2} = +\infty.$$

[10, Theorem 2.3, Corollary 6.11] or [4]. By the classical Julia-Caratheodory theorem (see [11]), this last condition is equivalent to nonexistence of the angular derivative of φ at any point $\xi \in \mathbb{T}$. The following theorem obtained in [5] generalizes the previous on Hardy and Bergman spaces .

Theorem 2.2. *Let ω be an admissible weight and $\varphi \in \mathcal{H}_\omega$. Then C_φ is compact on $\mathcal{H}_\omega$ if and only if*

$$\lim_{|z|\to 1^-} \frac{N_{\varphi,\omega}(z)}{\omega(z)} = 0 \tag{4}$$

In particular, Theorem 2.2 asserts that C_φ is compact on $\mathcal{D}_\alpha := \mathcal{H}_\alpha$ for $0<\alpha<1$ if and only if

$$N_{\varphi,\alpha}(z) := \sum_{\varphi(w)=z} (1-|w|^2)^\alpha = o((1-|z|^2)^\alpha).$$

The characterization of compact composition operators on the Dirichlet spaces $\mathcal{D}_\alpha$ in terms of Carleson measures for the Bergman spaces $\mathcal{A}^2_\alpha$ can be found in [3, 13, 16]. A positive Borel measure μ given on $\mathbb{D}$ is called a Carleson measure for the Bergman space $\mathcal{A}^2_\alpha$ if the identity map $i_\alpha : A^2_\alpha \to \mathrm{L}^2(\mu)$ is a bounded operator. Such a measure has the following equivalent properties (see [12] Theorem 1.2): A positive Borel measure μ is a Carleson measure for $\mathcal{A}^2_\alpha$ if and only if

$$\sup_{\lambda\in\mathbb{D}}(1-|\lambda|^2)^{2+\alpha}\int_{\mathbb{D}}\frac{d\mu(z)}{|1-\bar{\lambda}z|^{2(2+\alpha)}}<\infty \iff \sup_{I\subset\mathbb{T}}\frac{\mu(S(I))}{|I|^{2+\alpha}}<\infty,$$

for any subarc $I\subset\mathbb{T}$ with arclengh $|I|$, and the Carleson box

$$S(I)=\{z\in\mathbb{D}\colon\ z/|z|\in I,\ 1-|I|\le|z|<1\}.$$

The measure μ is called vanishing (or compact) Carleson measure for $\mathcal{A}^2_\alpha$ if the identity map $i_\alpha : \mathcal{A}^2_\alpha \to \mathrm{L}^2(\mu)$ is a compact operator. This happens if and only if

$$\lim_{|\lambda|\to1}(1-|\lambda|^2)^{2+\alpha}\int_{\mathbb{D}}\frac{d\mu(z)}{|1-\bar{\lambda}z|^{2(2+\alpha)}}=0 \iff \lim_{|I|\to0}\frac{\mu(S(I))}{|I|^{2+\alpha}}=0.$$

Therefore, as a consequence of Theorem 2.1 and Theorem 2.2,

$$\begin{aligned}
C_\varphi \quad \text{is bounded in } \mathcal{D}_\alpha \quad &\iff \sup_{I\subset\mathbb{T}}\frac{1}{|I|^{2+\alpha}}\int_{S(I)}N_{\varphi,\alpha}(z)\,dA(z)<\infty;\\
C_\varphi \quad \text{is compact in } \mathcal{D}_\alpha \quad &\iff \lim_{|I|\to0}\frac{1}{|I|^{2+\alpha}}\int_{S(I)}N_{\varphi,\alpha}(z)\,dA(z)=0.
\end{aligned}$$

That is C_φ is bounded on $\mathcal{D}_\alpha$ if and only if $N_{\varphi,\alpha}(z)dA(z)$ is a Carleson measure for $\mathcal{A}^2_\alpha$ and C_φ is compact on $\mathcal{D}_\alpha$ if and only if $N_{\varphi,\alpha}(z)dA(z)$ is a vanishing Carleson measure for $\mathcal{A}^2_\alpha$. Note that by the change of variable formula (1),

$$\begin{aligned}
J_\alpha(\lambda) \quad &:= \quad (1-|\lambda|^2)^{\alpha+2}\int_{\mathbb{D}}\frac{|\varphi'(z)|^2}{|1-\bar{\lambda}\varphi(z)|^{2(\alpha+2)}}\,dA_\alpha(z)\\
&= \quad (1-|\lambda|^2)^{2+\alpha}\int_{\mathbb{D}}\frac{N_{\varphi,\alpha}(z)\,dA(z)}{|1-\bar{\lambda}z|^{2(2+\alpha)}}.
\end{aligned}$$

Hence, we can also characterize, boundedness and compactness of C_φ on $\mathcal{D}_\alpha$, $0<\alpha<1$, by $\sup_{\lambda\in\mathbb{D}}J_\alpha(\lambda)<\infty$ and $\lim_{\lambda\to1}J_\alpha(\lambda)=0$. The Proposition 3.1 is devoted to the case where $\alpha=0$ corresponding to the classical Dirichlet space $\mathcal{D}$.

3. COMPOSITION OPERATORS ON THE DIRICHLET SPACE

We first give the following result, which was established in a general case in [13, 14, 15]. For the sake of completeness, we give here a simple proof

Proposition 3.1. *Let $\varphi : \mathbb{D} \to \mathbb{D}$ such that $\varphi \in \mathcal{D}$. Then*

(a) C_φ is bounded in $\mathcal{D}$ $\iff \sup_{\lambda\in\mathbb{D}}(1-|\lambda|^2)^2 \int_{\mathbb{D}} \frac{|\varphi'(z)|^2}{|1-\bar{\lambda}\varphi(z)|^4}\, dA(z) < \infty.$

(b) C_φ is compact in $\mathcal{D}$ $\iff \lim_{|\lambda|\to 1}(1-|\lambda|^2)^2 \int_{\mathbb{D}} \frac{|\varphi'(z)|^2}{|1-\bar{\lambda}\varphi(z)|^4}\, dA(z) = 0.$

Proof. (a). We only have to prove the converse; since the necessary condition easily follows from the fact that $z \to \dfrac{(1-|\lambda|^2)}{(1-\bar{\lambda}z)}$ is bounded (uniformly relatively to $\lambda \in \mathbb{D}$). Without loss of generality, we may assume that $\varphi(0) = 0$. Let $f \in \mathcal{D}$

$$\begin{aligned} \|C_\varphi(f)\|^2_{\mathcal{D}} &= |f(\varphi(0))|^2 + \int_{\mathbb{D}} |\varphi'(z)|^2 |f'(\varphi(z))|^2\, dA(z) \\ &\le |f(0)|^2 + \int_{\mathbb{D}} |\varphi'(z)|^2 \Big(\int_{\mathbb{D}} \frac{|f'(\lambda)|^2}{|1-\bar{\lambda}\varphi(z)|^4}\, dA(\lambda)\Big)\, dA(z) \\ &= |f(0)|^2 + \int_{\mathbb{D}} |f'(\lambda)|^2 \Big(\int_{\mathbb{D}} \frac{|\varphi'(z)|^2}{|1-\bar{\lambda}\varphi(z)|^4}\, dA(z)\Big)\, dA(\lambda) \\ &\le c\|f\|^2_{\mathcal{D}}. \end{aligned}$$

(b). We only have to prove the converse. Assume that the limit is equal to zero and let $(f_n)_n$ be a sequence which converges weakly to 0 in $\mathcal{D}$. Since $f'_n \to 0$ uniformly on compact sets, it follows the proof of part (a) and for r close to 1 that

$$\begin{aligned} \|C_\varphi(f_n)\|^2_{\mathcal{D}} - |f_n(0)|^2 \le& \int_{r\mathbb{D}} |f'_n(\lambda)|^2 (1-|\lambda|^2)^2 \int_{\mathbb{D}} \frac{|\varphi'(z)|^2}{|1-\overline{\lambda}\varphi(z)|^4}\, dA(z)\, dA(\lambda) \\ &+ \int_{\mathbb{D}\setminus r\mathbb{D}} |f'_n(\lambda)|^2 (1-|\lambda|^2)^2 \int_{\mathbb{D}} \frac{|\varphi'(z)|^2}{|1-\overline{\lambda}\varphi(z)|^4}\, dA(z)\, dA(\lambda) \to 0 \quad \text{as } n\to\infty. \end{aligned}$$

□

For $f \in \mathcal{D}$, the Dirichlet integral of f is given by

$$\mathcal{D}(f) = \int_{\mathbb{D}} |f'(z)|^2 dA(z).$$

The following result is an immediate consequence of Proposition 3.1.

Corollary 3.2. *Let $\varphi : \mathbb{D} \to \mathbb{D}$ such that $\varphi \in \mathcal{D}$.*

(a) If $\sup_{n\ge 1} \mathcal{D}(\varphi^n) < \infty$, then C_φ is bounded;

(b) If $\lim_{n\to\infty} \mathcal{D}(\varphi^n) = 0$*, then* C_φ *is compact.*

Proof. Both (a) and (b) follow from the following inequality:

$$\begin{aligned}(1-|\lambda|^2)^2 \int_{\mathbb{D}} \frac{|\varphi'(z)|^2}{|1-\overline{\lambda}\varphi(z)|^4}\, dA(z) &\le c(1-|\lambda|^2)^2 \int_{\mathbb{D}} \frac{|\varphi'(z)|^2}{(1-|\lambda|^2|\varphi(z)|^2)^4}\, dA(z) \\ &\le c\,(1-|\lambda|^2)^2 \sum_{n\ge 0} (n+1)^3\, |\lambda|^{2n} \int_{\mathbb{D}} |\varphi'(z)|^2 |\, \varphi^n(z)|^2\, dA(z) \\ &= c\,(1-|\lambda|^2)^2 \sum_{n\ge 0} (1+n)\, |\lambda|^{2n}\, \mathcal{D}(\varphi^{n+1}) \le c\, \limsup_{n\to\infty} \mathcal{D}(\varphi^{n+1}).\end{aligned}$$

□

We are interested herein in describing the spectral properties of the composition operator C_φ, such as compactness and Hilbert-Schmidt class membership, in terms of the size of the level set of φ. In order to state our second result, we introduce the notion of logarithmic capacity.

Given a (Borel) probability measure μ on $\mathbb{T}$, we define its energy by

$$I(\mu) = \sum_{n=1}^{\infty} \frac{|\widehat{\mu}(n)|^2}{n}.$$

For a closed set $E \subset \mathbb{T}$, its logarithmic capacity $\mathrm{cap}(E)$ is defined by

$$\mathrm{cap}(E) := 1/\inf\{I(\mu) : \mu \text{ is a probability measure on } E\}.$$

Since the Dirichlet space is contained in the Hardy space $\mathrm{H}^2(\mathbb{D})$, every function $f \in \mathcal{D}$ has non-tangential limits f almost everywhere on $\mathbb{T}$. In this case, however, more can be said. According to well-known result of Beurling [2], for each function $f \in \mathcal{D}$ then the radial limits $f(\zeta) = \lim_{r\to 1} f(r\zeta)$ exists q.e on $\mathbb{T}$, that is

$$\mathrm{cap}\big(\{\zeta \in \mathbb{T} : f(\zeta) \text{ does not exist}\}\big) = 0.$$

The weak–type inequality for capacity [2] states that, for $f \in \mathcal{D}$ and $t \ge 4\|f\|^2_{\mathcal{D}}$,

$$\mathrm{cap}(\{\zeta : |f(\zeta)| \ge t\}) \le \frac{16\|f\|^2_{\mathcal{D}}}{t^2}.$$

As a result of this inequality, we see that if $\liminf \|\varphi^n\|_{\mathcal{D}} = 0$, then $\mathrm{cap}(E_\varphi(1)) = 0$. Indeed, since $E_\varphi(1) = E_{\varphi^n}(1)$, the weak capacity inequality implies that

$$\mathrm{cap}(E_\varphi(1)) = \mathrm{cap}(E_{\varphi^n}(1)) \le 16\|\varphi^n\|^2_{\mathcal{D}}.$$

Now let $n \to \infty$, we can conclude. Hence, in particular, if the operator C_φ is in the Hilbert-Schmidt class in $\mathcal{D}$, $\sum_{n\ge 0} \|\varphi^n\|^2 < \infty$, then $\mathrm{cap}(E_\varphi(1)) = 0$. This result was first obtained by Gallardo-Gutiérrez and González [8] using a completely different method. Theorems 3.3 and 3.4 give quantitative versions of this result. The following theorem is similar to the Theorem 1.1 for the Dirichlet space

Theorem 3.3. *If C_φ is a Hilbert-Schmidt operator in $\mathcal{D}$, then*

$$\int_0^1 \frac{\operatorname{cap}(E_\varphi(s))}{1-s}\log\frac{1}{1-s}\,ds<\infty. \tag{5}$$

The following theorem shows that condition (5) is optimal.

Theorem 3.4. *Let $h:[1,+\infty[\to[1,+\infty[$ be a function such that $\lim_{x\to\infty}h(x)=+\infty$. Let E be a closed subset of $\mathbb{T}$ such that $\operatorname{cap}(E)=0$. Then there is $\varphi\in A(\mathbb{D})\cap\mathcal{D}$, $\varphi(\mathbb{D})\subset\mathbb{D}$ such that :*

(1) $E_\varphi(1)=E$;
(2) C_φ *is in the Hilbert–Schmidt class in* $\mathcal{D}$;
(3) $\displaystyle\int_0^1 \frac{\operatorname{cap}(E_\varphi(s))}{1-s}\log\frac{e}{1-s}h\Big(\frac{1}{1-s}\Big)\,ds=+\infty.$

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Compact weighted composition operators on some functional Hilbert spaces are also characterized. We give sufficient conditions for the compactness of such operators on weighted Dirichlet spaces. 1. Introduction A Hilbert space H(â„¦) of analytic functions on a domain â„¦ is called a functional Hilbert space provided the point evaluation f âˆ'â†' f (x) is continuous for every x in â„¦. The Hardy space H 2 and the Bergman space L2a (D) are the well-known examples of functional Hilbert spaces. An application of the Riesz representation theorem shows that for every x âˆˆ â„¦ there is a vector kx in H(â„¦) such th A weighted composition operator is a composition operator followed by a multipli-cation operator. More exactly, if Ïˆ is a complex function on E, then the transform. TÏˆ,Ï•f = MÏˆCÏ•f = Ïˆf â—¦ Ï• f âˆˆ S. is called the weighted composition operator of symbols Ïˆ and Ï•. Note that the rst symbol is that of the multiplication operator MÏˆ and the second that of the composition operator.Â Let us consider the Hilbert space L2a(D), the space of all analytic functions on D that are square integrable dA. Brennanâ€™s conjecture can be easily reformulated in terms of Ï„ = gâˆ'1. Indeed, elementary computations lead to the following equivalent formulation of Brennanâ€™s conjecture: If Ï„ is a Riemann transform of D onto a simply connected domain G C and âˆ'1/3 < p < 1, then 1/(Ï„)p âˆˆ L2a(D). composition operator on H{X) . This note is a report on the characterization of weighted composition operators on functional Hilbert spaces and the computation of the adjoint of such. 2 operators on L of an atomic measure space. Also the Fredholm criteria are discussed for such classes of operators. 1. Introduction. Let X be a non-empty set and l e t V(X) denote a Banach space of complex valued functions on X . Let T from i to X be a mapping such that for every / in V(X) , the composite function f Â° T is also in V(X) and the mapping C_ taking / to f Â° T is a bounded linear. operator on V(X) . Elliott, S.J., Wynn, A.: Composition operators on weighted Bergman spaces of a half-plane. Proc. Edinb.Â Kucik, A.S.: Carleson measures for Hilbert spaces of analytic functions on the complex half-plane. J. Math. Anal.Â Cite this article. Kucik, A.S. Weighted Composition Operators on Spaces of Analytic Functions on the Complex Half-Plane. Complex Anal. Oper.