K-Theory (2004) 33:325–346
DOI 10.1007/s10977-005-1554-7

Mapping Surgery to Analysis III: Exact Sequences

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(Received: February 2004)

Abstract. Using the constructions of the preceding two papers, we construct a natural transformation (after inverting 2) from the Browder–Novikov–Sullivan–Wall surgery exact sequence of a compact manifold to a certain exact sequence of C^*-algebra K-theory groups.

Mathematics Subject Classifications (1991): 19J25, 19K99.

Key words: C^*-algebras, L-theory, Poincaré duality, signature operator.

This is the final paper in a series of three whose objective is to construct a natural transformation from the surgery exact sequence of Browder, Novikov, Sullivan and Wall [17,21] to a long exact sequence of K-theory groups associated to a certain C^*-algebra extension; we finally achieve this objective in Theorem 5.4. In the first paper [5], we have shown how to associate a homotopy invariant C^*-algebraic signature to suitable chain complexes of Hilbert modules satisfying Poincaré duality. In the second paper, we have shown that such *Hilbert–Poincaré complexes* arise naturally from geometric examples of manifolds and Poincaré complexes. The C^*-algebras that are involved in these calculations are analytic reflections of the equivariant and/or controlled structure of the underlying topology. In paper II [6] we have also clarified the relationship between the *analytic signature*, defined by the procedure of paper I for suitable Poincaré complexes, and the *analytic index of the signature operator*, defined only for manifolds. In this paper we will complete the construction of our transformation from surgery to analysis, and it will turn out that the relationship between 'signatures' and 'signature operators' is fundamental to this construction. Briefly, to detect whether a homotopy equivalence of manifolds is a diffeomorphism, we may examine the mapping cylinder and ask whether this Poincaré space (with boundary) is in fact a manifold (with boundary). In turn, this question may be addressed analytically by asking whether a suitable analytic signature associated to the Poincaré space is actually the analytic index of some abstract elliptic operator.

The first section of this paper describes our analytic counterpart to the surgery exact sequence. In the second section we make a fundamental calculation relating coarse geometry and the boundary map in K-homology; this goes back to [3]. In the third section we use these ideas to construct a key invariant, the *structure invariant* of a Poincaré cobordism, and verify its properties. The fourth section contains a brief review of the surgery exact sequence. Finally, in the fifth section we use our structure invariant for Poincaré cobordisms to map the surgery exact sequence to an analytic exact sequence, and we verify the commutativity (up to powers of 2) of the resulting diagram.

1. The Analytic Surgery Sequence

Let π be a finitely generated group, fixed throughout the discussion. Recall from paper II, Definitions 2.6 and 2.8, that a *π-presented space* is a triple $(X, \widetilde{X}, \alpha)$ (usually abbreviated just to X) comprising a proper geodesic metric space $\widetilde{X}$, a free and proper action α of π on $\widetilde{X}$ by isometries, and a quotient space $X = \widetilde{X}/\pi$ with the induced geodesic metric. If X is a π-presented space, then an *(equivariant) X-module* is a Hilbert space H equipped with a nondegenerate representation ρ of $C_0(\widetilde{X})$ and a compatible unitary representation of π. For future reference we remark that any representation ρ of $C_0(\widetilde{X})$ on a Hilbert space extends canonically to a representation (still denoted ρ) of the algebra of bounded Borel functions on $\widetilde{X}$; non-degeneracy means that the extended representation is unital.

Now recall the basic definitions of C^*-algebraic coarse geometry.

DEFINITION 1.1. Let X be a π-presented space. The category $\mathfrak{D}^*(X)$ has objects the (equivariant) X-modules and morphisms the norm limits of equivariant, finite propagation operators $T\colon H \to H'$ between them which are *pseudolocal*, that is, which satisfy

$$T\rho(f) - \rho'(f)T \in \mathfrak{K}(H, H')$$

for every $f \in C_0(\widetilde{X})$.

DEFINITION 1.2. Let X be a π-presented space. The category $\mathfrak{C}^*(X)$ has objects the (equivariant) X-modules and morphisms the norm limits of equivariant, finite propagation operators $T\colon H \to H'$ between them which are *locally compact*, that is, which satisfy

$$T\rho(f) \in \mathfrak{K}(H, H'), \quad \rho'(f)T \in \mathfrak{K}(H, H')$$

for every $f \in C_0(\widetilde{X})$.

The category $\mathfrak{C}^*(X)$ was already defined in paper II (Definition 2.10). The category $\mathfrak{D}^*(X)$ is a subcategory of the category $\mathfrak{A}^*(X)$ defined in paper II.

The quotient of a C^*-category by an ideal can be defined, and it is again a C^*-category. For our categories we have the following important result.

PROPOSITION 1.3 (Paschke Duality theorem). *Let X be a π-presented space. For each i there is a natural isomorphism*

$$K_{i+1}(\mathfrak{D}^*(X)/\mathfrak{C}^*(X)) \cong K_i(X).$$

The group appearing on the right is the Kasparov K-homology of X, that is, the group $KK^{-i}(C_0(X), \mathbb{C})$. Because π acts freely and properly on $\widetilde{X}$ this is in fact the same thing as the equivariant Kasparov group $KK_\pi^{-i}(C_0(\widetilde{X}), \mathbb{C})$.

Proof. In [4] (compare also [7, Chapter 8]), the Paschke duality theorem is proved in the following form: $K_{i-1}(X) = K_i(\Psi^0(X)/\Psi^{-1}(X))$, where $\Psi^0(X)$ is the category of pseudolocal operators on *unequivariant* X-modules, and Ψ^{-1} is the ideal of locally compact operators. To obtain the form of Paschke duality given in the proposition we must therefore show that

$$\Psi^0(X)/\Psi^{-1}(X) \cong \mathfrak{D}^*(X)/\mathfrak{C}^*(X),$$

in other words, we must gain analytic control and equivariance.

Call a subset $U \subseteq X$ *elementary* if its inverse image $\pi^{-1}(U)$ (under the covering projection $\pi\colon \widetilde{X} \to X$) is identified with $U \times \pi$; every point of X has an elementary neighborhood. Let H be an unequivariant X-module, which we may assume is of the form $L^2(X, \mu)$, and let $\widetilde{H} = L^2(\widetilde{X}, \tilde{\mu})$ be the corresponding equivariant module (see Remark 2.9 of paper II). Let $\{U_i\}$ be a locally finite cover of X by elementary open sets of bounded diameter, $\leqslant r$ say, and let $\{\varphi_i^2\}$ be a subordinate partition of unity. Because each U_i is elementary there is an identification

$$\widetilde{H}_i := L^2(\pi^{-1}(U_i), \tilde{\mu}) = L^2(U_i, \mu) \otimes \ell^2(\pi).$$

Using this identification lift the operator $T_i = \varphi_i T \varphi_i$ on $L^2(U, \mu)$ to an operator $\widetilde{T}_i = T_i \otimes 1$ on $\widetilde{H}_i \subseteq \widetilde{H}$, and finally define

$$\Phi(T) := \sum T_i \in \mathfrak{B}(\widetilde{H}).$$

The map $\Phi\colon \mathfrak{B}(H) \to \mathfrak{B}(\widetilde{H})$ is linear and completely positive, and its image consists of equivariant operators of propagation $\leqslant r$.

Introduce the notation $\Psi^0(H)$ for the algebra of pseudolocal operators on H, and define $\Psi^{-1}(H)$, $D^*(\widetilde{H})$, $C^*(\widetilde{H})$ similarly. It is easily verified that

Φ maps $\Psi^0(H)$ to $D^*(\widetilde{H})$ and $\Psi^{-1}(H)$ to $C^*(\widetilde{H})$. Thus we obtain a linear map

$$\Psi^0(H)/\Psi^{-1}(H) \to D^*(\widetilde{H})/C^*(\widetilde{H}).$$

The proof is then completed by noting the following three facts, all of whose proofs are simple calculations:

(a) the linear map displayed above is injective;
(b) the linear map displayed above is surjective;
(c) the linear map displayed above is a $*$-homomorphism.

In connection with (c), note that while Φ itself is of course not a $*$-homomorphism, it is a $*$-homomorphism 'modulo compacts' in an appropriate sense. □

Remark 1.4. The original reference for Paschke duality is [12].

Let X be a π-presented space. From the short exact sequence of C^*-categories

$$0 \to \mathfrak{C}^*(X) \to \mathfrak{D}^*(X) \to \mathfrak{D}^*(X)/\mathfrak{C}^*(X) \to 0$$

together with Paschke duality, one obtains a 'long' (six-term, thanks to Bott periodicity) exact sequence of K-theory groups. Using Paschke duality, we may express this exact sequence as

$$\cdots \to K_{i+1}(\mathfrak{D}^*(X)) \to K_i(X) \to K_i(\mathfrak{C}^*(X)) \to K_i(\mathfrak{D}^*(X)) \to \cdots$$

DEFINITION 1.5. We call this exact sequence the *analytic surgery exact sequence* associated to the π-presented space X. The map $\mu_X\colon K_i(X) \to K_i(\mathfrak{C}^*(X))$ appearing in this sequence is called the *analytic assembly map*.

Remark 1.6. If X is compact, then (as we observed in paper II), the category $\mathfrak{C}^*(X)$ is Morita equivalent to the group C^*-algebra $C^*_r(\pi)$. Thus the analytic assembly map becomes a map $K_i(X) \to K_i(C^*_r(\pi))$. In this case it coincides with other formulations of the assembly map, such as that appearing in the Baum-Connes conjecture (see [2] for the conjecture and [19] for the comparison of the assembly maps).

In certain situations the assembly map μ_X appearing above is conjectured, or even proved, to be an isomorphism. In these cases $K_*(\mathfrak{D}^*(X)) = 0$, and conversely the non-vanishing of $K_*(\mathfrak{D}^*(X))$ is an obstruction to μ_X being an isomorphism. For this reason it is of some interest to devise means of constructing elements in this K-theory group. *Our objective in this* *paper is to relate the analytic surgery sequence for a compact manifold X to the usual surgery sequence of high-dimensional topology, and in particular to produce a natural map from the manifold structure set $\mathcal{S}(X)$ to $K_*(\mathfrak{D}^*(X))$.* The topologist's surgery sequence will be reviewed in Section 4. See Chapter 7 of [18] for a brief account of the construction detailed here, as well as for a different method of producing elements of $K_*(\mathfrak{D}^*(X))$ (using positive scalar curvature metrics).

Remark 1.7. Suppose that X is a complete oriented Riemannian n-manifold. The *signature operator* D_X of X is an elliptic operator and therefore, by the usual machinery of K-homology theory, defines a class $[D_X] \in K_n(X)$. By definition, the assembly map μ_X takes this homology class to the (coarse) *index* $\mathtt{Index}(D_X) \in K_n(\mathfrak{C}^*(X))$ (see Section 12.3 of [7]). On the other hand, the machinery of paper I defines an *analytic signature* $\mathtt{Sign}(X) \in K_n(\mathfrak{C}^*(X))$. By Theorem 5.5 of paper II, we have

$$\mu_X([D_X]) = \mathtt{Index}(D_X) = \mathtt{Sign}(X).$$

2. The Analytic Surgery Sequence and the Boundary Map in K-Homology

Even though our ultimate objective involves the analytic surgery sequence for compact manifolds only, we will need in the construction to consider *non-compact* spaces of various sorts.

DEFINITION 2.1. Let Y be a compact manifold. A *conelike end* $\mathcal{O}Y$ based on Y is a product manifold $Y \times \mathbb{R}^+$ equipped with a (complete) Riemannian metric of the form

$$ds^2 = dt^2 + \varphi(t)^2 g_{ij} dy^i dy^j,$$

where t is the coordinate on $\mathbb{R}^+$, g_{ij} is a Riemannian metric on Y and $\varphi\colon \mathbb{R}^+ \to \mathbb{R}^+$ is a C^∞ function such that $\varphi(t) = 1$ for sufficiently small t and $\varphi(t) = t$ for sufficiently large t.

If $\pi_1(Y) = \pi$, we may regard a conelike end based on Y as a π-presented space (by considering its universal covering). We will say that a π-presented space X *has a conelike end based on* Y if there is a closed geodesically convex subset of $\widetilde{X}$ which is π-equivariantly isometric to the universal cover of a conelike end $\mathcal{O}(Y)$ as defined above.

Remark 2.2. It is important to observe that a conelike end is a Riemannian manifold of *bounded geometry*. It can therefore be equipped with a smooth triangulation of bounded geometry (the general results of Calabi alluded to in paper II are not needed to see this; one can construct the required triangulation directly).

If $\mathcal{O}(Y)$ is a conelike end we can *compactify* it by adding a copy of Y at infinity. The resulting space is diffeomorphic to $Y\times[0,1]$. Similarly, if X is a π-presented space with a conelike end based on Y we can *compactify the end* by adding a copy of Y at infinity. We denote the resulting space by X_Y, and we equip it with any geodesic metric which agrees with the original metric on X outside of a compact (in X_Y) convex neighborhood of the boundary space Y. (Since we are only interested in coarse geometry, any two such metrics will be equivalent for our purposes.) The space X_Y is again a π-presented space. (Beware that X_Y need not be compact, since X may have other ends besides the conelike one that we are considering.)

Every function in $C_0(\widetilde{X_Y})$ restricts to a bounded Borel, indeed continuous, function on $\widetilde{X}$. Thus, every equivariant X-module automatically carries the structure of an equivariant X_Y-module.

LEMMA 2.3. *Let X be a π-presented space with a conelike end based on Y, and let X_Y be obtained by compactifying the end, as above. Let H, H' be equivariant X-modules and let $T\colon H\to H'$ be a morphism in $\mathfrak{D}^*(X)$. Then T is also a morphism in $\mathfrak{D}^*(X_Y)$ when H, H' are considered as equivariant X_Y-modules. Thus, there is a morphism of C^*-categories*

$$\beta\colon \mathfrak{D}^*(X)\to\mathfrak{D}^*(X_Y).$$

Proof. This result originates from [3], and has appeared in various places in the literature, for instance in [18, Chapter 10]; for completeness we sketch a few details. It suffices to show that if T has finite propagation and is pseudolocal for X, then it is also pseudolocal for X_Y. An important lemma of Kasparov states that T is pseudolocal if and only if

$$\rho'(f')T\rho(f)\in\mathfrak{K}(H,H')$$

for all $f, f'\in C_0(\widetilde{X_Y})$ having disjoint supports. Since f and f' have disjoint supports in X_Y, the supports of their restrictions to $\mathcal{O}Y$ 'diverge at infinity'; for any $r>0$ there is a neighborhood of infinity, $U\subseteq\mathcal{O}Y$ such that the distance from $\mathtt{Supp}(f)\cap U$ to $\mathtt{Supp}(f')$ and the distance from $\mathtt{Supp}(f')\cap U$ to $\mathtt{Supp}(f)$ are greater than r. Take r equal to the propagation of T and let $g, g'\in C_0(X)$ agree with f, f' on $X\backslash U$. Then $\rho(f')T\rho(f)=\rho(g')T\rho(g)$ and this is compact by the originally given pseudolocality of T. □

Composing the homomorphism of Lemma 2.3 with the quotient map from $\mathfrak{D}^*(X_Y)$ to $\mathfrak{D}^*(X_Y)/\mathfrak{C}^*(X_Y)$, and then applying Paschke duality, we obtain a homomorphism

$$b\colon K_i(\mathfrak{D}^*(X)) \to K_{i-1}(X_Y).$$

DEFINITION 2.4. We will call the homomorphism b appearing above the *descent homomorphism* associated to the conelike end of X.

There is also a version of this construction for $\mathfrak{C}^*(X)$ rather than $\mathfrak{D}^*(X)$. To define it, choose a neighborhood of infinity U on the conelike end, as in the previous proof. We can extend continuous functions on $\widetilde{Y}$ to bounded Borel functions on $\widetilde{X}$ by first extending to U using the product structure of the end, and then extending by zero outside U. This process makes every equivariant X-module into an equivariant Y-module, and the same proof as before shows that we obtain a homomorphism of C^*-categories $\mathfrak{C}^*(X) \to \mathfrak{D}^*(Y)$. The choice of cut-off U affects this homomorphism only by an element of $\mathfrak{C}^*(Y)$ and we therefore finally obtain a well-defined homomorphism

$$\mathfrak{C}^*(X) \to \mathfrak{D}^*(Y)/\mathfrak{C}^*(Y)$$

yielding another K-theoretic descent homomorphism

$$b\colon K_i(\mathfrak{C}^*(X)) \to K_{i-1}(Y).$$

PROPOSITION 2.5. *Let X be a π-presented space with a conelike end based on Y, and let X_Y be obtained by compactifying the end, as above. The descent homomorphisms described above fit into a commutative diagram of exact sequences*

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & K_i(X) & \longrightarrow & K_i(\mathfrak{C}^*(X)) & \longrightarrow & K_i(\mathfrak{D}^*(X)) & \longrightarrow & \cdots \\
 & & \downarrow{=} & & \downarrow{b} & & \downarrow{b} & & \\
\cdots & \longrightarrow & K_i(X) & \longrightarrow & K_{i-1}(Y) & \longrightarrow & K_{i-1}(X_Y) & \longrightarrow & \cdots
\end{array}$$

in which the top row is the analytic surgery exact sequence and the bottom row is the K-homology exact sequence of the pair (X_Y, Y).

Proof. This depends on the interpretation of the K-homology long exact sequence in terms of Paschke duality, which is reviewed in Chapter 5 of [7]. In our context we may express this as follows. Since $C_0(X) \subseteq C_0(X_Y)$, every X_Y-module restricts to an X-module. This gives a restriction homomorphism

$$\mathfrak{D}^*(X_Y)/\mathfrak{C}^*(X_Y) \to \mathfrak{D}^*(X)/\mathfrak{C}^*(X).$$

It may be shown that this restriction homomorphism is surjective (this is the 'Excision Theorem' for K-homology). Its kernel is known as the 'relative dual' algebra of Y in X_Y; we will denote it by $\mathfrak{R}(Y, X_Y)$. There is a natural inclusion

$$\mathfrak{D}^*(Y)/\mathfrak{C}^*(Y) \to \mathfrak{R}(Y, X_Y)$$

and this inclusion induces an isomorphism on K-theory (Proposition 5.3.7 of [7]). Thus the long exact sequence of K-theory groups associated to the short exact sequence

$$0 \longrightarrow \mathfrak{R}(Y, X_Y) \longrightarrow \mathfrak{D}^*(X_Y)/\mathfrak{C}^*(X_Y) \longrightarrow \mathfrak{D}^*(X)/\mathfrak{C}^*(X) \longrightarrow 0$$

is the K-homology exact sequence of the pair (X_Y, Y).

Now we see that the descent maps of the proposition arise from homomorphisms of C^*-categories that fit into a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathfrak{C}^*(X) & \longrightarrow & \mathfrak{D}^*(X) & \longrightarrow & \mathfrak{D}^*(X)/\mathfrak{C}^*(X) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathfrak{R}(Y, X_Y) & \longrightarrow & \mathfrak{D}^*(X_Y)/\mathfrak{C}^*(X_Y) & \longrightarrow & \mathfrak{D}^*(X)/\mathfrak{C}^*(X) & \longrightarrow & 0 \end{array}$$

and the result follows when we consider the corresponding diagram of K-theory long exact sequences. □

We shall need one result about the behavior of analytic signatures under descent homomorphisms associated to conelike ends. Let (X, Y) be a finite Poincaré pair, and let Y_i be the connected components of Y. Recall from surgery theory [21] that (X, Y) satisfies the *π–π condition* if each inclusion $Y_i \to X$ induces an isomorphism on fundamental groups. (We allow X itself to be disconnected, but in that case we insist that the fundamental groups of all the components of X should be isomorphic.)

DEFINITION 2.6. A *good Poincaré space* is a geometrically controlled Poincaré complex W obtained as follows:

(a) Start with a finite Poincaré pair (X, Y) which satisfies the π–π condition and all of whose boundary components are smooth manifolds Y_i;
(b) Attach either a conelike end or a cylinder-like end ($Y \times \mathbb{R}^+$ with product metric) to each boundary component.

Remark 2.7. W is a geometrically controlled Poincaré complex in the category of π-presented spaces, where $\pi = \pi_1(X) = \pi_1(W)$. We are making use of a geometrically controlled version of the result [21, Proposition 2.7] that one can glue two Poincaré pairs along a common boundary to obtain a

Poincaré space. Since (X, Y) is finite, it is geometrically controlled (Lemma 3.15 of paper II); and since the conelike ends are bounded geometry manifolds, they also are geometrically controlled. The gluing maps are geometrically controlled since they are defined over the compact space Y.

A good Poincaré space W has an analytic signature $\mathtt{Sign}(W)$ living in the K-theory group $K_n(\mathfrak{C}^*(W))$.

Remark 2.8. We will also need a bordism invariance result for the signatures of good Poincaré spaces. Let us say that a *bordism of good Poincaré spaces* is a Poincaré bordism which, outside a compact set, is (isometrically) a product $\coprod_i Z_i \times I$, where each Z_i is a conelike or cylinder-like end. Such a bordism gives rise to a geometrically controlled Poincaré pair. It follows from Theorem 3.17 of paper II and Theorem 7.9 of paper I that bordant good Poincaré spaces have the same signature.

THEOREM 2.9. *Let W be a good Poincaré space, with a conelike end based on Y. Let $b\colon K_n(\mathfrak{C}^*(W)) \to K_{n-1}(Y)$ be the descent homomorphism associated to the conelike end. Then*

$$b[\mathtt{Sign}(W)] = k_n[D_Y] \in K_{n-1}(Y),$$

where the constant k_n equals 1 if n is odd and 2 if n is even.

Proof. First, the result is true if the good Poincaré space W is in fact a complete Riemannian manifold. In that case, by Remark 1.7, the signature $\mathtt{Sign}(W) \in K_n(\mathfrak{C}^*(W))$ is the image of the signature operator class $[D_W]$ under the assembly map. By the commutativity of the diagram

$$\begin{array}{ccc} K_n(W) & \longrightarrow & K_n(\mathfrak{C}^*(W)) \\ \big\downarrow = & & \big\downarrow b \\ K_n(W) & \longrightarrow & K_{n-1}(Y) \end{array},$$

from Proposition 2.5, the image $b[\mathtt{Sign}(W)]$ is therefore equal to the image of $[D_W]$ under the K-homology boundary map $K_n(W) = K_n(W_Y, Y) \to K_{n-1}(Y)$. It is well known, however, that the boundary of the homology class of the signature operator is k_n times the homology class of the signature operator on the boundary.

The remainder of the proof therefore consists of a reduction to the manifold case. Let W' be the complete Riemannian manifold $Y \times \mathbb{R}$, with a metric which is conelike on the positive end of $\mathbb{R}$ and is a product on the negative end. Thus W' is another good Poincaré space with a conelike end based on Y, and $b[\mathtt{Sign}(W')] = k_n[D_Y]$. We will construct two good Poincaré spaces X_1 and X_2, equipped with coarse maps φ_1 and φ_2 to W', such that

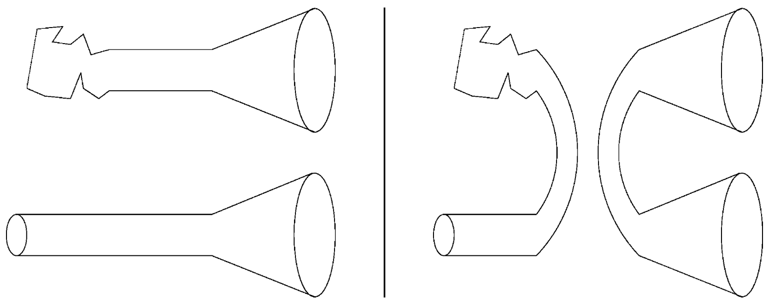

Figure 1. The space X_1 (left figure) and the space X_2 (right figure).

(a) $b[\varphi_{1*}(\mathtt{Sign}(X_1))] = b[\mathtt{Sign}(W)] - b[\mathtt{Sign}(W')] \in K_{n-1}(Y)$;
(b) $b[\varphi_{2*}(\mathtt{Sign}(X_2))] = 0 \in K_{n-1}(Y)$;
(c) (X_1, φ_1) and (X_2, φ_2) are bordant (in the sense of Remark 2.8).

Since bordant spaces have the same signature (Remark 2.8), it will follow from (a), (b) and (c) that $b[\mathtt{Sign}(W)] = b[\mathtt{Sign}(W')] = k_n[D_Y]$, proving the result.

The construction is illustrated in Fig. 1. The space X_1 is the disjoint union of W and $-W'$ (where $-W'$ denotes W' with reversed orientation). It is equipped with the coarse map φ_1 to W' which is the identity on W' and on the conelike end of W, and which on the remainder of W sends a point w to $(y_0, -d(w, w_0)) \in Y \times \mathbb{R}^-$, y_0 and w_0 being fixed base points in Y and W respectively. The space X_2 is identical with X_1 except on a compact region; on this compact region it has been changed by a bordism so as to replace $Y \times S^0 \times B^1$ by $Y \times B^1 \times S^0$. The map φ_2 agrees with φ_1 outside the compact set, and is arbitrary inside it. Items (a) and (c) above are now clear (the minus sign in (a) comes from the reversal of the orientation on W'). As for (b), the left-hand component of X_2 makes no contribution to image of the signature after descent to the end. Thus we may replace X_2 by its right-hand component alone, which is again a Riemannian manifold. We find that $b[\mathtt{Sign}(X_2)]$ is the image of $k_n([D_Y] \oplus (-[D_Y])) \in K_{n-1}(Y \sqcup Y)$ under the folding map $Y \sqcup Y \to Y$. This image is obviously zero. □

3. The Structure Invariant

In this section V will denote a closed n-dimensional manifold with fundamental group π. By a *Poincaré cobordism over* V we will mean an $(n+1)$-dimensional Poincaré pair $(W, \partial W)$ such that

(i) The boundary ∂W is a disjoint union $\partial_- W \sqcup \partial_+ W$, where $\partial_\pm W$ are closed n-manifolds;

(ii) The triple $(W; \partial_- W, \partial_+ W)$ is equipped with a map f (of triples) to $(V \times [0,1]; V \times \{0\}, V \times \{1\})$, whose restriction to each boundary component is smooth;

(iii) The map f is of degree one and it induces an isomorphism on fundamental groups; moreover, the same is true of each of its restrictions to maps $f_\pm \colon \partial_\pm W \to V$.

Such a Poincaré cobordism will be denoted briefly (W, f), or just W if the map f can be understood.

There is an evident notion of *addition* of such Poincaré cobordisms. Indeed, if W' and W'' are two such cobordisms, with $\partial_+ W' = -\partial_- W''$ (in which case we will say that W' and W'' are *compatible*; the notation should also be taken to imply the agreement of the maps f', f'' on the respective boundary components) then we may glue them to form a cobordism

$$W = W' \cup_{\partial_+ W' = \partial_- W''} W''$$

which is equipped with an evident map $f = f' \cup f''$ to $V \times [0,2]$. We will write $W = W' + W''$ in these circumstances.

In this section we will associate to any Poincaré cobordism (W, f) over V a *structure invariant* $\sigma(W, f) \in K_{n+1}(\mathfrak{D}^* V)$, which has the following properties:

(a) (Vanishing) If (W, f) is a *manifold* cobordism, $\sigma(W, f) = 0$;

(b) (Additivity) If (W, f) and (W', f') are compatible Poincaré cobordisms, then

$$\sigma((W', f') + (W, f)) = \sigma(W', f') + \sigma(W, f).$$

(c) (Signature) The image of $\sigma(W, f)$ under the map $K_{n+1}(\mathfrak{D}^* V) \to K_n(V)$ appearing in the analytic surgery exact sequence is (up to a constant) the difference of the signature classes. More precisely, it equals

$$k_{n+1}(f_*[D_{\partial_+ W}] - f_*[D_{\partial_- W}]),$$

where k_{n+1} equals one if n is even, and equals two if n is odd.

(d) (Homotopy) $\sigma(W, f)$ is unchanged by homotopy equivalences (keeping the boundary fixed) of the cobordism (W, f).

Let (W, f) be a Poincaré cobordism. Form a good Poincaré space $\widehat{W}$ by attaching conelike ends (Definition 2.1) to the two boundary components $W_\pm$ of W, so that

$$\widehat{W} = \mathcal{O}\partial_- W \cup W \cup \mathcal{O}\partial_+ W$$

with suitable choice of orientation. By construction, $\widehat{W}$ comes with a continuous coarse map $\hat{f}$ to the open double cone $\mathcal{B}V$, which is just $V \times \mathbb{R}$ with a metric which is conical on both ends. Notice also that all spaces we have mentioned are π-presented spaces in a natural way.

Remark 3.1. It is important for this construction that the boundary components of our cobordism should be *manifolds*. The open cone on a non-manifold Poincaré space is usually not a bounded geometry Poincaré space in a natural way.

Arising from the geometry of $\mathcal{B}V$ we have the following series of maps:

$$\begin{aligned} K_{n+1}(\mathfrak{C}^*(\mathcal{B}V)) \to K_{n+1}(\mathfrak{D}^*(\mathcal{B}V)) \to K_{n+1}(\mathfrak{D}^*(V \times [0,1])) \\ \to K_{n+1}(\mathfrak{D}^*(V)). \end{aligned} \tag{3.1}$$

The first map is induced by inclusion. The second map comes from compactifying the conelike ends of $\mathcal{B}V$ to obtain $V \times [0,1]$, together with Lemma 2.3. The third map is functorially induced by the projection $V \times [0,1] \to V$.

DEFINITION 3.2. We let $\chi_V \colon K_{n+1}(\mathfrak{C}^*(\mathcal{B}V)) \to K_{n+1}(\mathfrak{D}^*(V))$ denote the composite of the maps in the sequence (3.1) above.

DEFINITION 3.3. The *structure invariant* $\sigma(W, f) \in K_{n+1}(\mathfrak{D}^*(V))$ is the image, under the composite map χ_V of Definition 3.2 above, of $\hat{f}_*(\mathtt{Sign}\,\widehat{W})$, where $\mathtt{Sign}\,\widehat{W} \in K_{n+1}(\mathfrak{C}^*(\widehat{W}))$ is the signature of the Hilbert–Poincaré space $\widehat{W}$.

THEOREM 3.4. *The structure invariant has the vanishing, signature, additivity, and homotopy properties listed above as (a)–(d).*

Proof. Consider first the vanishing property (a). Suppose that W is in fact a *manifold* cobordism. Then $\widehat{W}$ is a complete Riemannian manifold, and thus by Remark 1.7 the signature $\mathtt{Sign}\,\widehat{W}$ is equal to the image of the homology class $[D_{\widehat{W}}] \in K_{n+1}(\widehat{W})$ of the signature operator under the assembly map $\mu \colon K_{n+1}(\widehat{W}) \to K_{n+1}(\mathfrak{C}^*(\widehat{W}))$. Hence (because f is continuous), $\hat{f}_*(\mathtt{Sign}\,\widehat{W})$ belongs to the image of the assembly map for $\mathcal{B}V$, and so maps to zero in $K_{n+1}(\mathfrak{D}^*(\mathcal{B}V))$ by exactness in the analytic surgery sequence for $\mathcal{B}V$.

Now consider the additivity property (b). Let (W', f') and (W'', f'') be two compatible Poincaré cobordisms, with sum (W, f). Let $\widehat{W}$, and so on, denote the spaces obtained by coning these cobordisms, as above. Now consider the following two good Poincaré spaces equipped with the obvious maps to $\mathcal{B}V$:

$$A = \widehat{W'} \sqcup \widehat{W''}, \quad B = \widehat{W} \sqcup \mathcal{B}M$$

where M denotes $\partial_+ W' = \partial_- W''$. The space B is obtained from A by cutting off one end and gluing it back on the other way around, so that in fact A and B are bordant as good Poincaré spaces. It follows (Remark 2.8) that the signature of A and the signature of B are equal in the group $K_*(\mathfrak{C}^*(\mathcal{B}V))$. However, the signature of A maps, under the composite map 3.1, to $\sigma(W')+\sigma(W'')$, whereas the signature of B maps to $\sigma(W)+\sigma(W_0)$, where W_0 is the trivial cobordism $M\times[0,1]$. Since $\sigma(W_0)=0$ by the vanishing property, we have proved the result.

Let us check the signature property (c). Associated to the conelike ends of $\widehat{W}$ there is a descent map

$$b\colon K_{n+1}(\mathfrak{C}^*(\widehat{W}))\to K_n(\partial_- W\sqcup\partial_+ W).$$

By Theorem 2.9, $b(\mathtt{Sign}\,W)=k_{n+1}([D_{\partial_- W}]-[D_{\partial_+ W}])$. The asserted result now follows by chasing through the following diagram:

$$\begin{array}{ccccc}
K_{n+1}(\mathfrak{C}^*(\widehat{W})) & \longrightarrow & & \longrightarrow & K_n(\partial_- W\sqcup\partial_+ W) \\
\downarrow \hat{f} & & & & \downarrow (f_-,f_+) \\
K_{n+1}(\mathfrak{C}^*(\mathcal{B}V)) & \longrightarrow & & \longrightarrow & K_n(V\sqcup V) \\
\downarrow & & & & \downarrow \\
K_{n+1}(\mathfrak{D}^*(\mathcal{B}V)) & \longrightarrow & K_{n+1}(\mathfrak{D}^*(V\times[0,1])) & \longrightarrow & K_n(V\times[0,1]) \\
 & & \downarrow & & \downarrow = \\
 & & K_{n+1}(\mathfrak{D}^*V) & \longrightarrow & K_n(V)
\end{array}$$

in which the three leftmost horizontal arrows are various descent maps.

Finally, the homotopy property (d) of our invariant $\sigma(W,f)$ follows immediately from the homotopy invariance of the Hilbert–Poincaré signature. □

4. Review of Surgery Theory

In this section we give a very brief review of the surgery exact sequence. The reader may consult Wall's book [21], or more recent expositions such as [17,22], for further information about the following notions.

Let V be a smooth, closed, oriented, n-dimensional manifold. The central object of attention in surgery theory is the *structure set* $\mathcal{S}(V)$. The members of $\mathcal{S}(V)$ are equivalence classes of *homotopy manifold structures* on V, by which we mean orientation-preserving homotopy equivalences $f\colon M\to V$, where M is a smooth manifold. Two such structures $f_i\colon M_i\to V$ $(i=0,1)$ are declared to be equivalent if they are *h-cobordant*, meaning that there is a cobordism W between M_0 and M_1, and W itself is equipped with a homotopy equivalence $f\colon W\to M$ which restricts on the boundary components to the given maps f_i. In particular, two structures are equivalent if they are *diffeomorphic* in the sense that there is a homotopy commuting diagram

$$\begin{array}{ccc} M_0 & \xrightarrow{\;h\;} & M_1 \\ & {\scriptstyle f_0}\searrow \quad \swarrow{\scriptstyle f_1} & \\ & V & \end{array}$$

with h a diffeomorphism.

Remark 4.1. The structure set that we have defined is sometimes denoted $\mathcal{S}^h(V)$, to distinguish it from an alternative version, $\mathcal{S}^s(V)$, in which the homotopy equivalences are required to be *simple*. The 'simple homotopy' version of surgery is more directly related to the diffeomorphism classification of the manifolds, because of the s-cobordism theorem [9]: two high-dimensional manifolds are s-cobordant if and only if they are diffeomorphic. However, this refinement is not needed for the construction of our maps; and in fact it is known that the two versions of surgery differ only by 2-torsion.

Surgery theory embeds the structure set in the *surgery exact sequence*. The surgery exact sequence (or at least the portion of it that we propose to study) is

$$L_{n+1}(\pi) \cdots\cdots\cdots > \mathcal{S}(V) \longrightarrow \mathcal{N}(V) \longrightarrow L_n(\pi)$$

with $\pi = \pi_1 V$. We need to recall the definition of the L groups and of the normal invariant terms $\mathcal{N}(V)$ in this sequence.

$\mathcal{N}(V)$ is the set of degree one normal maps $f\colon M \to V$, modulo the equivalence relation defined by normal bordism. A 'normal map' is a map that is covered by an equivalence of Spivak normal bundles [21, Chapter 1]. However for the purposes of our construction it is not necessary to know the details of this 'bundle data'; the normal bordism set maps forgetfully to ordinary (oriented) bordism $\Omega_n(V)$, and our map from surgery to analysis will factor through this forgetful map.[1]

The abelian group $L_m(\pi)$ is defined geometrically, following [21, Chapter 9]. It is also a cobordism group. Specifically, a *cycle* for $L_m(\pi)$ consists of a degree one normal map of pairs $f\colon (W, \partial W) \to (X, \partial X)$, where $(X, \partial X)$ is an m-dimensional oriented Poincaré pair and f restricts to a homotopy equivalence on the boundaries. Also given as part of the cycle is a map $X \to B\pi$. On these cycles is imposed a cobordism relation which we will

[1]This is because we will tensor with $\mathbb{Z}[\frac{1}{2}]$.

not describe just yet (the reader will find the details in [21]). The *realization theorem* states that every element of the group $L_{n+1}(\pi)$, $\pi = \pi_1(V^n)$, has a representative of a particularly simple kind:

THEOREM 4.2. *Let V be a closed manifold of dimension n and having fundamental group π, and let $f: M \to V$ be a manifold structure on V. Then any element $\alpha \in L_{n+1}(\pi)$ has a representative which is a map $h: W \to V \times$* $[0, 1]$, *where W is a cobordism, $\partial_- W = M$ and $h_{|\partial_- W} = f$, and $h_{|\partial_+ W}$ gives another manifold structure on V.*

We can now describe the maps appearing in the surgery sequence. The map $\mathcal{S}(V) \to \mathcal{N}(V)$ is the natural forgetful map: an orientation-preserving homotopy equivalence from a manifold to V is certainly a degree one normal map. The map $\mathcal{N}(V)$ to $L_n(\pi)$ is also a forgetful map: a normal map $f: M \to V$ can be considered as a normal map of pairs (with empty boundary), and this together with the map $V \to B\pi$ which classifies the universal cover gives a cycle for $L_n(\pi)$. Exactness at $\mathcal{N}(V)$ is the fundamental theorem of surgery theory: a degree one normal map is normal bordant to a homotopy equivalence if and only if its surgery obstruction vanishes. Finally we must describe the dotted arrow from $L_{n+1}(\pi)$ to $\mathcal{S}(V)$. This arrow denotes an action of the group $L_{n+1}(\pi)$ on the set $\mathcal{S}(V)$, and its exactness means that two structures belong to the same orbit of this action if and only if they have the same normal invariant. The action is defined by Wall realization (Theorem 4.2): given a structure $f: M \to V$ in $\mathcal{S}(V)$ and an element $x \in L_{n+1}(\pi)$, realization provides a representative for $-x$ of the form $h: W \mapsto V \times [0, 1]$, with the homotopy equivalence h_0 equal to f; then $f + x$ is defined to be the structure represented by h_1.

5. Mapping Surgery to Analysis

In this section we will define our maps from surgery to analysis, and then will prove that the resulting diagram of exact sequences is commutative.

Let V be a smooth, closed, oriented, n-dimensional manifold, with fundamental group π. Here is the diagram that we will be considering (we will refer to this as the 'Main Diagram').

$$
\begin{array}{ccccccc}
L_{n+1}(\pi) & \dashrightarrow & \mathcal{S}(V) & \longrightarrow & \mathcal{N}(V) & \longrightarrow & L_n(\pi) \\
\downarrow{\scriptstyle \gamma} & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \gamma} \\
K_{n+1}(C_r^*\pi) \otimes \mathbb{Z}[\frac{1}{2}] & \to & K_{n+1}(\mathfrak{D}^*(V)) \otimes \mathbb{Z}[\frac{1}{2}] & \to & K_n(V) \otimes \mathbb{Z}[\frac{1}{2}] & \to & K_n(C_r^*\pi) \otimes \mathbb{Z}[\frac{1}{2}]
\end{array}
$$

The bottom row of this diagram is the analytic surgery sequence for V (Definition 1.5 and Remark 1.6), tensored with $\mathbb{Z}[\frac{1}{2}]$.

DEFINITION 5.1. We define the map $\alpha\colon \mathcal{S}(V) \to K_{n+1}(\mathfrak{D}^*(V)) \otimes \mathbb{Z}[\frac{1}{2}]$. Let $f\colon M \to V$ be a structure on V, that is a homotopy equivalence between V and a manifold M. We may regard f as a homotopy equivalence between M and V as manifolds over V, in other words, we may consider the commutative diagram

$$\begin{array}{ccc} M & \xrightarrow{f} & V \\ & {\scriptstyle f}\searrow \quad \swarrow{\scriptstyle =} & \\ & V & \end{array}$$

Let C_f be the mapping cylinder $M \cup_f V \times [0, 1]$, which is a Poincaré cobordism (in fact an h-cobordism) over V. We define $\alpha[f] = 2^{-\lfloor (n+1)/2 \rfloor}\sigma(C_f)$, where $\sigma(C_f) \in K_{n+1}(\mathfrak{D}^*(V))$ is the structure invariant of Definition 3.3, and $\lfloor x \rfloor$ denotes the greatest integer $\leqslant x$.

To show that α is well-defined, suppose that f and f' are two structures related by a manifold h-cobordism W. Then $C_f + W$ is homotopy equivalent (rel boundary) to $C_{f'}$, with the notion of addition of cobordisms defined in the previous section. However, $\sigma(W) = 0$ by the vanishing property (Theorem 3.4(a)), since W is a *manifold* cobordism. Therefore $\sigma(C_f) = \sigma(C_{f'})$ by the additivity and homotopy properties (Theorem 3.4(b,d)), and the map α is well-defined.

DEFINITION 5.2. We define the map β: $\mathcal{N}(V) \to K_n(V) \otimes \mathbb{Z}[\frac{1}{2}]$. Let $f\colon M \to V$ be a degree one normal map; then the map β sends f to the 'difference of signature operators' of M and V. Specifically, $\beta[f] = 2^{-\lfloor n/2 \rfloor}(f_*[D_M] - [D_V]) \in K_n(V) \otimes \mathbb{Z}[\frac{1}{2}]$, where $[D]$ denotes the K-homology class of the signature operator.

To show that β is well-defined, let $h\colon W \to V \times [0, 1]$ be a normal cobordism. It will be enough to prove that the signature operator $D_{\partial W}$ maps to zero in $K_n(V) \otimes \mathbb{Z}[\frac{1}{2}]$ (under the map $h_0 \sqcup h_1\colon \partial W \to V$). But, up to a multiple of 2, $D_{\partial W}$ belongs to the image of the boundary map $K_{n+1}(W, \partial W) \to K_n(\partial W)$ in K-homology, and the result follows from this via the commutative diagram

$$\begin{array}{ccccc} & K_{n+1}(W, \partial W) & \longrightarrow & K_n(\partial W) & \\ & \downarrow & & \downarrow & \\ K_{n+1}(V \times [0,1], V \times \{0,1\}) & \longrightarrow & K_n(V \times \{0,1\}) & \longrightarrow & K_n(V \times [0,1]) \\ & & \downarrow & \swarrow & \\ & & K_n(V) & & \end{array}$$

in which the middle row is exact.

DEFINITION 5.3. We define the map $\gamma\colon L_n(\pi)\to K_n(C_r^*(\pi))\otimes\mathbb{Z}[\frac{1}{2}]$. Let $f\colon (M,\partial M)\to(X,\partial X)$ be a degree one normal map of pairs, in which X is a Poincaré space of formal dimension n, f is a homotopy equivalence on the boundary, and there is given a map $X\to B\pi$. (These data make up a cycle x for $L_n(\pi)$.) Glue M to X (reversing the orientation of the latter) by means of the homotopy equivalence of their boundaries, obtaining a compact Poincaré space Z (without boundary), equipped with a map to $B\pi$. Now define $\gamma[M,X,f]=2^{-\lfloor n/2\rfloor}\,\mathtt{Sign}\,Z$, where $\mathtt{Sign}\,Z\in K_n(\mathfrak{C}^*(Z))=K_n(C_r^*(\pi))$ is the analytic signature of the Poincaré duality space Z.

To check that γ is well-defined, one must come to grips with the definition in [21, Chapter 9] of the equivalence relation that gives rise to the group $L_n(\pi)$. By definition, an object of the sort considered above (that is, a degree one normal map $f\colon (M,\partial M)\to(X,\partial X)$) is equivalent to zero if there is a degree one normal map of triads[2] $g\colon (W,\partial_1 W,\partial_2 W)\to(Y,\partial_1 Y,\partial_2 Y)$ such that

(a) $(\partial_1 W,\partial_{12}W)=(M,\partial M)$, $(\partial_1 Y,\partial_{12}Y)=(X,\partial X)$, and the restriction of g to $\partial_1 W$ agrees with f.
(b) The restriction of g to $\partial_2 W$ is a homotopy equivalence.
(c) The given map $X\to B\pi$ extends to Y.

If such a bordism exists then we may glue W to Y along ∂_2 using the homotopy equivalence g; the resulting space is a compact Poincaré pair whose boundary is the space Z of the definition above. Thus $\mathtt{Sign}(Z)=0$ by the bordism invariance of the analytic signature (Remark 4.8 of paper II). It follows that γ is well defined.

Here is the main result of our paper.

THEOREM 5.4. *The Main Diagram (displayed at the beginning of this section) is commutative.*

Proof. We check the commutativity of the right-hand square. Let $f\colon M\to V$ be a degree one normal map defining an element of $\mathcal{N}(V)$. The corresponding L-theory element is defined by the same normal map f, now thought of as a map of pairs with empty boundary. Thus, the space Z appearing in Definition 5.3 is the disjoint union $M\sqcup(-V)$, and thus $\gamma[M,V,f]=2^{-\lfloor n/2\rfloor}(\mathtt{Sign}\,M-\mathtt{Sign}\,V)$. On the other hand, by definition $\beta[f]=2^{-\lfloor n/2\rfloor}(f_*[D_M]-[D_V])$. Since the homology class of the signature operator passes under assembly to the analytic signature (Remark 1.7), we see that the right-hand square commutes.

We check the commutativity of the middle square. Let $f\colon M\to V$ be a structure on V. The corresponding invariant in $K_{n+1}(\mathfrak{D}^*(V))\otimes\mathbb{Z}[\frac{1}{2}]$

[2]See [21], again, for this notion.

is $\alpha[f] = 2^{-\lfloor (n+1)/2 \rfloor}\sigma(C_f)$, where C_f is the mapping cylinder of f. By Theorem 3.4(c), the image of $\alpha[f]$ under the map $K_{n+1}(\mathfrak{D}^*(V)) \otimes \mathbb{Z}[\frac{1}{2}] \to K_n(V) \otimes \mathbb{Z}[\frac{1}{2}]$ is

$$k_{n+1} 2^{-\lfloor (n+1)/2 \rfloor}(f_*[D_M] - [D_V]).$$

But $k_{n+1} 2^{-\lfloor (n+1)/2 \rfloor} = 2^{-\lfloor n/2 \rfloor}$ so the displayed expression above is equal to $\beta[f]$.

Finally we check the commutativity of the left-hand square. (This commutativity means that the action of $L_{n+1}(\pi)$ on $\mathcal{S}(V)$ given by Wall realization is compatible with the natural affine action of $K_{n+1}(C_r^*(\pi)) = K_{n+1}(\mathfrak{C}^*(V))$ on $K_{n+1}(\mathfrak{D}^*(V))$.) Let $x \in L_{n+1}(\pi)$ and a structure $f_0 \colon M \to V$ be given. By Wall realization (Theorem 4.2) we may construct a (manifold) cobordism $f \colon W \to V \times [0, 1]$ with f_0 the given structure, f_1 a new structure, and surgery obstruction x; then, by definition, $f_1 = f_0 + x$. We want to prove that $\sigma(C_{f_1}) = \sigma(C_{f_0}) + i_*(\gamma(x))$, where $i \colon \mathfrak{C}^*(V) \to \mathfrak{D}^*(V)$ is the canonical inclusion.

Form a Poincaré cobordism P from V to itself over V by gluing the mapping cylinders of f_0 and f_1 onto the boundary pieces of the cobordism W (reversing the orientation of the mapping cylinder of f_0). We claim that

$$\sigma(P) = i_*(\gamma(x)) \tag{5.1}$$

If this claim is granted it is easy to complete the proof; by the additivity property of the structure invariant (Theorem 3.4(b)), we have

$$\sigma(P) = -\sigma(C_{f_0}) + \sigma(W) + \sigma(C_{f_1}) = -\sigma(C_{f_0}) + \sigma(C_{f_1}),$$

$\sigma(W)$ vanishes as W is a *manifold* cobordism (Theorem 3.4(a)).

It remains to check the claim expressed by Equation 5.1. According to Definition 5.3, the class $\gamma(x) \in K_{n+1}(\mathfrak{C}^* V)$ is the signature of the Poincaré space Z which is obtained from P by identifying its two boundary components (both of which are copies of V). Z is naturally equipped with a map to V which we regard as $V \times \{0\}$ inside the double cone $\mathcal{B}(V)$ (see section 3 for the definition of this double cone and its use in the construction of the structure invariant). Under the map χ_V of Definition 3.2, the signature of Z maps to $i_*(\gamma(x))$.

Consider now the disjoint union $Q = Z \sqcup \mathcal{B}(V)$ as a good Poincaré space over $\mathcal{B}(V)$. This space has a signature $\mathtt{Sign}(Q) = \mathtt{Sign}(Z) + \mathtt{Sign}(\mathcal{B}(V))$. We have $\chi_V(\mathtt{Sign}(\mathcal{B}(V))) = 0$ (since $\mathtt{Sign}(\mathcal{B}(V))$ is in the image of the assembly map) and so $\chi_V(\mathtt{Sign}(Q)) = \chi_V(\mathtt{Sign}(Z)) = i_*(\gamma(x))$.

Finally note that Q_1 is bordant over $\mathcal{B}(V)$ to the space $\widehat{P}$ obtained by attaching conelike ends to P. Thus

$$\chi_V(\mathtt{Sign}(Q)) = \chi_V(\mathtt{Sign}(\widehat{P}) = \sigma(P)$$

by definition of the structure invariant and bordism invariance. Thus we finally obtain $\sigma(P) = i_*(\gamma(x))$ as required. □

6. Examples and Discussion

It is not hard to find examples where our structure invariant is non-zero. First consider the simply-connected case with $n = \dim V$ divisible by 4. Then we may identify the assembly map $K_0(V) \to K_0(\mathfrak{K}) = \mathbb{Z}$ with the map induced on K-homology by crushing V to a point (the index map). The analytic structure set $K_1(\mathfrak{D}^*(V))$ is therefore identified with the *reduced* K-homology $\widetilde{K}_0(V)$ of V. If we use the Chern character to make the (rational) identification

$$K_0(V) \otimes \mathbb{Q} = H_{ev}(V; \mathbb{Q})$$

then the signature operator class on V passes to the Poincaré dual of the Hirzebruch L-class, and similarly the structure invariant of $f: M \to V$ corresponds to the difference of the dual L-classes of M and V (both these classes have the same image in $H_0(\mathrm{pt})$ by the Hirzebruch signature theorem, so their difference does belong to reduced homology). Now it is well-known that, within the homotopy type of a simply-connected manifold, the only universal constraint on the rational Pontrjagin classes is that provided by the signature theorem (see [8]); we conclude that, at least rationally, any element of $\widetilde{H}_{n-4j}(V; \mathbb{Q})$, $j \geqslant 1$, can be realized as an analytic structure invariant.

In the non-simply-connected case there are further examples. A positive linear functional $\tau: C_r^*(\pi) \to \mathbb{C}$ is called a *normalized trace* if $\tau(1) = 1$ and $\tau(xy) = \tau(yx)$ for all $x, y \in C_r^*(\Gamma)$. (There is a canonical example, given by taking the coefficient of the identity element, but there may be others; for instance, any finite-dimensional representation of $C_r^*(\Gamma)$ will give rise to a normalized trace arising from the standard trace on matrix algebras.) The following is a version of the L^2 index theorem [1]:

THEOREM 6.1. *Let V be a compact even-dimensional oriented manifold with fundamental group π. Then for any normalized trace τ on $C_r^*(\pi)$, the induced homomorphism*

$$\tau_*: K_0(C_r^*(\pi)) \to \mathbb{R}$$

maps $\mu_V[D]$ to the index of D, for any elliptic operator D on V. □

Suppose now that $\tau = (\tau_1, \ldots, \tau_k)$ is a list of normalized traces on $C_r^*(\pi)$. It defines a homomorphism $K_0(C_r^*(\pi)) \to \mathbb{R}^k$, and, by the theorem, this homomorphism maps $\mathrm{Im}\,\mu$ to the diagonally embedded copy of $\mathbb{R}$ in $\mathbb{R}^k$. Thus

$$\tau_*: K_0(C_r^*(\pi)) \to \mathbb{R}^k/\mathbb{R}$$

detects the cokernel of the assembly map μ, which is a subgroup of the analytic structure set. When π is finite and the traces arise from finite-dimensional representations, this analytic invariant corresponds to[3] the *multisignature* of [21, Chapter 13]. Thus exotic manifold structures which are detected by the multisignature, such as some fake lens spaces, will also be detected by our analytic structure invariant.

Remark 6.2. One could contemplate carrying out the constructions of this paper in the context of KO-theory (the K-theory of real C^*-algebras), rather than (as we have done) in the context of complex K-theory. This however would seem to require a different treatment of the signature of a Hilbert–Poincaré complex, distinguishing four or eight cases.

Remark 6.3. Our constructions have produced a map from the *smooth* surgery exact sequence to our 'analytic surgery' sequence. But surgery can also be carried out in other categories; and in fact the surgery theory of *topological* manifolds (see [10, 15, 16] for discussion) is most closely related to K-theory. One might therefore ask whether it is possible to construct a diagram similar to our own relating analysis to *topological* surgery. To see that this is a non-trivial task, note that the existence of such a diagram would immediately imply that the K-homology classes of the signature operators on homeomorphic smooth manifolds are equal (modulo 2-torsion, at least); and this statement, in turn, immediately implies Novikov's theorem on the topological invariance of the rational Pontrjagin classes [11]. Thus an 'analytic proof of Novikov's theorem' would be implicit in the construction of a topological version of our result.

Following a question of Singer [20], such analytic proofs have long been sought; but the situation is rather unsatisfactory. The necessary constructions can be carried out algebraically, but they do not *a priori* respect the requirement that the operators involved be *bounded* in the appropriate L^2 norms; so it is not clear whether the constructions can be carried out in the category of Hilbert spaces and bounded operators. In [14] this problem is referred to as the problem of 'gaining analytic control'; it is circumvented by an excision argument, which shows for homological reasons that (over appropriate control spaces) the two versions of the theory, one using bounded operators and one ignoring that restriction, must agree. It seems possible that a similar argument could be combined with our construction to produce a map from $\mathcal{S}^{TOP}(X)$ to $K_*(\mathfrak{D}^*(X))$. In fact, one can describe

[3] Strictly speaking, Wall's multisignature also contains additional information relating to possible real or quaternionic structures on the finite-dimensional representations being considered.

the fiber of the assembly map – essentially the TOP structure set – as the L-theory of a certain controlled category (see Pedersen [13] for one way of doing this). If one 'completes' this category to a C^*-category in the obvious way it should be possible to construct a map from the K-theory of the resulting C^*-category to $K_*(\mathfrak{D}^*(X))$; and so, if we can solve the problem of gaining analytic control in the Pedersen category, we will obtain the desired map. But this is a discussion for a different paper.

Acknowledgements

The authors were supported in part by NSF Grant DMS-0100464.

References

1. Atiyah, M. F.: Elliptic operators, discrete groups and von Neumann algebras, *Astérisque*, **32** (1976), 43–72.
2. Baum, P., Connes, A. and Higson, N.: Classifying space for proper G-actions and K-theory of group C^*-algebras, in *Proceedings of a Special Session on C^*-Algebras*, *Contemporary Mathematics*, Vol. 167, American Mathematical Society, Providence, RI (1994), pp. 241–291.
3. Higson, N.: On the relative K-homology theory of Baum and Douglas, Unpublished preprint, 1990.
4. Higson, N.: C^*-algebra extension theory and duality, *J. Funct. Anal.* **129** (1995), 349–363.
5. Higson, N. and Roe, J.: Mapping surgery to analysis I: analytic signatures, this issue.
6. Higson, N. and Roe, J.: Mapping surgery to analysis II: geometric signatures, this issue.
7. Higson, N. and Roe, J.: Analytic K-Homology. *Oxford Mathematical Monographs*, Oxford University Press, Oxford, 2000.
8. Kahn, P. J.: A note on topological Pontrjagin classes and the Hirzebruch index formula, *Illinois J. Math.* **16** (1972), 243–256.
9. Kervaire, M.: Le thèoréme de Barden-Mazur-Stallings, *Comment. Math. Helvetici*, **40** (1965), 31–42.
10. Kirby, R. C. and Siebenmann, L. C.: Foundational Essays on Topological Manifolds, Smoothings, and Triangulations, *Annals of Mathematics Studies*, No. 88, Princeton University Press, Princeton, NJ (1976).
11. Novikov, S. P.: The topological invariance of the rational Pontrjagin classes, *Math. USSR Doklady* **6** (1965), 921–923.
12. Paschke, W. L.: K-theory for commutants in the Calkin algebra, *Pacific J. Math.* (95) (1981), 427–437.
13. Pedersen, E. K.: The surgery exact sequence revisited, in: *High-Dimensional Manifold Topology*, World Scientific, River Edge, NJ (2003), pp. 416–420.
14. Pedersen, E. K., Roe, J. and Weinberger, S.: On the homotopy invariance of the boundedly controlled analytic signature of a manifold over an open cone, in: S. Ferry, A. Ranicki, and J. Rosenberg (eds.), *Proceedings of the 1993 Oberwolfach Conference on the Novikov Conjecture*, *LMS Lecture Notes*, Vol. 227, Cambridge University Press, Cambridge (1995), pp. 285–300.
15. Quinn, F.: $BTOP_n$ and the surgery obstruction, *Bull. AMS* **77** (1971), 596–600.

16. Ranicki, A.: *Algebraic L-Theory and Topological Manifolds*, Cambridge University Press, Cambridge (1992).
17. Ranicki, A.: *Algebraic and Geometric Surgery*, Clarendon Press, Oxford (2002).
18. Roe, J.: *Index Theory, Coarse Geometry, and the Topology of Manifolds*, *CBMS Conference Proceedings*, Vol. 90, American Mathematical Society, Providence, RI (1996).
19. Roe, J.: Comparing analytic assembly maps, *Oxford Quart. J. Math.* **53** (2002), 1–8.
20. Singer, I. M.: Future extensions of index theory and elliptic operators, in: *Prospects in Mathematics*, *Annals of Mathematics Studies*, Vol. 70, Princeton University Press, Princeton, NJ (1971), pp. 171–185.
21. Wall, C. T. C.: *Surgery on Compact Manifolds*, Academic Press, Boston (1970).
22. Weinberger, S.: *The Topological Classification of Stratified Spaces*, University of Chicago Press, Chicago (1995).

It provides tools for flu sequence analysis, annotation and submission to GenBank. This resource also has links to other flu sequence resources, and publications and general information about flu viruses. Downloads. BLAST (Stand-alone).Â Finds regions of local similarity between biological sequences. The program compares nucleotide or protein sequences to sequence databases and calculates the statistical significance of matches. BLAST can be used to infer functional and evolutionary relationships between sequences as well as to help identify members of gene families. COBALT. We construct an isomorphism between the geometric model and Higson-Roeâ€™s analytic surgery group, reconciling the constructions in the previous papers.Â Following work of Piazza and Schick, we construct a geometric map from Stolzâ€™ positive scalar curvature sequence to the geometric model of Higson-Roeâ€™s analytic surgery exact sequence. This is a preview of subscription content, access via your institution. Access options. Buy single article. Instant access to the full article PDF. 34,95 â‚¬. Tax calculation will be finalised during checkout. Subscribe to journal. Immediate online access to all issues from 2019. Subscription will auto renew annually. 111,21 â‚¬. Printed in the Netherlands. 325. Mapping Surgery to Analysis III: Exact Sequences. NIGEL HIGSON and JOHN ROE Department of Mathematics, Penn State University, University Park, Pennsylvania 16802. e-mail: roe@math.psu.edu; higson@math.psu.edu. (Received: February 2004). Abstract.Â This is the nal paper in a series of three whose objective is to construct a natural transformation from the surgery exact sequence of Browder, Novikov, Sullivan and Wall [17,21] to a long exact sequence of K-theory groups associated to a certain Câˆ—-algebra extension; we nally achieve this objective in Theorem 5.4. Higson, Nigel ; Roe, John. / Mapping surgery to analysis III : Exact sequences. In: K-Theory. 2004 ; Vol. 33, No. 4. pp. 325-346. @article{1fe5242203464f73be28087eff4459d4, title = "Mapping surgery to analysis III: Exact sequences", abstract = "Using the constructions of the preceding two papers, we construct a natural transformation (after inverting 2) from the Browder-Novikov-Sullivan-Wall surgery exact sequence of a compact manifold to a certain exact sequence of C *-algebra K-theory groups.", author = "Nigel Higson and John Roe"Â Research output: Contribution to journal â€º Article â€º peer-review. Ty - jour. T1 - Mapping surgery to analysis III. T2 - Exact sequences. AU - Higson, Nigel. AU - Roe, John. Py - 2004/12/1. Y1 - 2004/12/1.