Matrices over Integral Domains
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1 Introduction

In this article we present some results on matrices over integral domains, which extend the well known results for matrices over the fields discussed in §1 of this book. The general theory of linear algebra over commutative rings is extensively studied the the book [McD84]. It is mostly intended for readers with a thorough training in ring theory. The aim of this article is to give a brief survey of notions and facts about matrices over classical domains that come up in applications. Namely over the ring of integers, the ring of polynomials over the field, the ring of analytic functions in one variable on an open connected set, and germs of analytic functions in one variable at the origin. The last section of this article is devoted to the notion of strict equivalence of pencils.

Most of the results in this article are well known to the experts. A few new results are taken from the book in progress [Frixx], which are mostly contained in the preprint [Fri81].

2 Certain Integral Domains

In this section we discuss properties of certain integral domains. Most of the standard facts about domains used here can be found in [ZS58]. More special results and references on the elementary divisor domains and rings are in [McD84]. The standard results on the domains of analytic functions can be found in [GuR65]. More special results on analytic functions in one complex variable are in [Rud74].

Definitions:
A commutative ring without zero divisors and containing identity 1 is called an **integral domain** and denoted by $\mathbb{D}$.

The **quotient field** F of a given integral domain $\mathbb{D}$ is formed by the set of equivalence classes of all quotients $\frac{a}{b}, b \neq 0$, where $\frac{a}{b} \equiv \frac{c}{d}$ if and only if $ad = bc$, such that

$$\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}, \quad \frac{a}{b}\frac{c}{d} = \frac{ac}{bd}, \quad b, d \neq 0.$$

For $\mathbf{x} = (x_1, \ldots, x_n)^T \in \mathbb{D}^n, \alpha = (\alpha_1, \ldots, \alpha_n)^T \in \mathbb{Z}_+^n$ we define $\mathbf{x}^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ and $|\alpha| = \sum_{i=1}^n |\alpha_i|$.
$\mathbb{D}[x_1, \ldots, x_n] = \mathbb{D}[\mathbf{x}]$ is the ring of all polynomials $p(\mathbf{x})$ in n variables with coefficients in $\mathbb{D}$ $p(\mathbf{x}) = \sum_{|\alpha| \leq m} a_\alpha \mathbf{x}^\alpha$ for some $m \in \mathbb{Z}_+$.
The **total degree**, or simply the **degree** of $p(\mathbf{x}) \neq 0$, denoted by deg p, is $m \in \mathbb{Z}_+$ if there exists $a_\alpha \neq 0$ such that $|\alpha| = m$. (deg $0 = -\infty$.)
A polynomial p is called **homogeneous** if $a_\alpha = 0$ for all $|\alpha| <$ deg p.
A polynomial $p(x) = \sum_{i=0}^n a_i x^i \in \mathbb{D}[x]$ is called **monic** if $a_n = 1$.
$F(\mathbf{x})$ denotes the quotient field of $F[\mathbf{x}]$, and is called the **field of rational functions** over F in n variables.
Let $\Omega \subset \mathbb{C}^n$ be a nonempty path-connected set. Then $\mathrm{H}(\Omega)$ denotes the ring of analytic functions $f(\mathbf{z})$, such that for each $\zeta \in \Omega$ there exists an open neighborhood $O(\zeta, f)$ of ζ such that f is analytic on $O(f, \zeta)$. The addition and the product of functions are given by the standard identities: $(f+g)(\zeta) = f(\zeta)+g(\zeta), (fg)(\zeta) = f(\zeta)g(\zeta)$. If Ω is an open set we assume that f is defined only on Ω. If Ω consists of one point ζ then H_ζ stands for $\mathrm{H}(\{\zeta\})$.
Denote by $\mathcal{M}(\Omega), \mathcal{M}_\zeta$ the quotient fields of $\mathrm{H}(\Omega), \mathrm{H}_\zeta$ respectively.
For $a, d \in \mathbb{D}$, d **divides** a, (or d is a **divisor** of a), denoted by $d|a$, if $a = db$ for some $b \in \mathbb{D}$.
$a \in \mathbb{D}$ is called **unit** if $a|1$.
$a, b \in \mathbb{D}$ are called **associates**, which is denoted by $a \equiv b$, if $a|b$ and $b|a$. Denote $\{\{a\}\} := \{b \in \mathbb{D} : \ b \equiv a\}$.
The associates of $a \in \mathbb{D}$ and the units are called **improper** divisors of a.
$a \in \mathbb{D}$ is called **irreducible** if it is not a unit and every divisor of a is improper.
A nonzero, nonunit element $p \in \mathbb{D}$ is called **prime** if for any $a, b \in \mathbb{D}, p|ab$ implies $p|a$ or $p|b$.
Let $a_1, \ldots, a_n \in \mathbb{D}$. Assume first that not all of $a_1, \ldots, a_n$ are equal to zero. An element $d \in \mathbb{D}$ is called a **greatest common divisor** (g.c.d) of $a_1, ..., a_n$ if $d|a_i$ for $i = 1, ..., n$, and for any d' such that $d'|a_i, i = 1, ..., n$, $d'|d$. Denote by $(a_1, ..., a_n)$ any g.c.d. of $a_1, \ldots, a_n$. Then $\{\{(a_1, \ldots, a_n)\}\}$ is the equivalence class of all g.c.d. of $a_1, \ldots, a_n$. For $a_1 = \ldots = a_n = 0$ we define 0 to be the g.c.d. of $a_1, \ldots, a_n$, i.e. $(a_1, \ldots, a_n) = 0$.
$a_1, ..., a_n \in \mathbb{D}$ are called **coprime** if $\{\{(a_1, ..., a_n)\}\} = \{\{1\}\}$.
$I \subseteq \mathbb{D}$ is called an **ideal** if for any $a, b \in I$ and $p, q \in \mathbb{D}$ the element $pa + qb$ belongs to I.
I is called **prime** if $ab \in I$ implies that either a or b is in I.
I is called **maximal** if the only ideals which contain I is I and $\mathbb{D}$.
I is called **finitely generated** if there exists k elements (**generators**) $p_1, ..., p_k \in I$ such that any $i \in I$ is of the form $i = a_1p_1 + \cdots + a_kp_k$ for some $a_1, ..., a_k \in \mathbb{D}$.
An ideal is called **principal** ideal if it is generated by one element p.

$\mathbb{D}$ is called a **greatest common divisor** domain (GCDD), denoted by $\mathbb{D}_g$, if any two elements in $\mathbb{D}$ have a g.c.d..
$\mathbb{D}$ is called a **unique factorization** domain (UFD), denoted by $\mathbb{D}_u$, if any nonzero, nonunit element a can be factored as a product of irreducible elements $a = p_1 \cdots p_r$, and this factorization is unique within order and unit factors.
$\mathbb{D}$ is called a **principal ideal** domain (PID), denoted $\mathbb{D}_p$, if any ideal of $\mathbb{D}$ is principal.
$\mathbb{D}$ is called a **Euclidean** domain (ED), denoted $\mathbb{D}_e$, if there exists a function $d : \mathbb{D}\backslash\{0\} \to \mathbb{Z}_+$ such that:

$$\begin{aligned}&\text{for all } a, b \in \mathbb{D},\ ab \neq 0 \quad d(a) \leq d(ab);\\ &\text{for any } a, b \in \mathbb{D},\ ab \neq 0,\ \text{there exists } t, r \in \mathbb{D} \text{ such that}\\ &a = tb + r,\ \text{where either } r = 0 \text{ or } d(r) < d(b).\end{aligned}$$

It is convenient to define $d(0) = \infty$. Let $a_1, a_2 \in \mathbb{D}_e$ and assume that $\infty > d(a_1) \geq d(a_2)$. The **Euclid's Algorithm** consists of a sequence $a_1, ..., a_{k+1}$, where $(a_1 \dots a_k) \neq 0$, which is defined recursively as follows:

$$a_i = t_i a_{i+1} + a_{i+2}, \quad a_{i+2} = 0 \text{ or } d(a_{i+2}) < d(a_{i+1}) \text{ for } i = 1, \dots k-1.$$

[Hel43, Kap49] $\mathbb{D}$ is called an **elementary divisor** domain (EDD), denoted by $\mathbb{D}_{ed}$, if for any three elements $a, b, c \in D$ there exists $p, q, x, y \in D$ such that $(a, b, c) = (px)a + (py)b + (qy)c$.
A GCDD is called a **Bezout** domain (BD), denoted by $\mathbb{D}_b$, if for any two elements $a, b \in \mathbb{D}$ $(a, b) = pa + qb$, for some $p, q \in \mathbb{D}$.
$p(x) = \sum_{i=0}^{m} a_i x^{m-i} \in \mathbb{Z}[x], a_0 \neq 0, m \geq 1$ is called **primitive** if 1 is a g.c.d. of $a_0, \dots, a_m$.
For $m \in \mathbb{N}$, the set of integers modulo m is denoted by $\mathbb{Z}_m$.

Facts:

1. Any integral domain satisfies **cancelations laws**: if $ab = ac$ or $ba = ca$ and $a \neq 0$ then $b = c$.

2. An integral domain such that any nonzero element is unit is a field F, and any field is an integral domain in which any nonzero element is unit.

3. A finite integral domain is a field.

4. $\mathbb{D}[\mathbf{x}]$ is an integral domain.

5. H(Ω) is an integral domain.

6. Any prime element in $\mathbb{D}$ is irreducible.

7. In UFD any irreducible element is prime. This is not true in all integral domains.

8. Let $\mathbb{D}$ is UFD. Then $\mathbb{D}[x]$ is UFD. Hence $\mathbb{D}[\mathbf{x}]$ is UFD.

9. Let $a_1, a_2 \in \mathbb{D}_e$ and assume that $\infty > d(a_1) \geq d(a_2)$. Then Euclid's Algorithm terminates in a finite number of steps, i.e. there exists $k \geq 3$ such that $a_1 \neq 0, \ldots, a_k \neq 0$ and $a_{k+1} = 0$. Hence $a_k = (a_1, a_2)$.

10. ED is PID.

11. PID is EDD.

12. PID is UFD.

13. EDD is BD.

14. BD is GCDD.

15. UFD is GCDD.

16. the converse of the above six implications is false.

17. An integral domain is a Bezout domain if and only if any finitely generated ideal is principal.

18. $\mathbb{Z}$ is ED with $d(a) = |a|$.

19. Let $p, q \in \mathbb{Z}[x]$ be primitive polynomials. Then pq is primitive.

20. $F[x]$ is ED with $d(p)$ - the degree of a nonzero polynomial. Hence $\mathbb{F}[\mathbf{x}]$ is UFD.

21. $\mathbb{Z}[x_1, ..., x_m]$, $F[x_1, ..., x_n]$ and $\mathrm{H}(\Omega), \Omega \subseteq \mathbb{C}^n$ are GCDD, but for $m \geq 1$ and $n \geq 2$ these domains are not BD.

22. [Frixx], (see Example 17). Let $\Omega \subset \mathbb{C}$ be an open connected set. Then for $a, b \in \mathrm{H}(\Omega)$ there exists $p \in \mathrm{H}(\Omega)$ such that $(a, b) = pa + b$.

23. For a connected set $\Omega \subset \mathbb{C}$ $\mathrm{H}(\Omega)$ is GCDD.

24. H_ζ, $\zeta \in \mathbb{C}$, is UFD.

25. If $\Omega \subset \mathbb{C}^n$ be a connected open set then $\mathrm{H}(\Omega)$ is not UFD. (For $n = 1$ there is no prime factorization of an analytic function $f \in \mathrm{H}(\Omega)$ with an infinite countable number of zeros.)

26. Let $\Omega \subset \mathbb{C}$ be a compact connected set. Then $\mathrm{H}(\Omega)$ is ED. Here $d(a)$ is the number of zeros of a nonzero function $a \in \mathrm{H}(\Omega)$ counted with their multiplicities.

27. [Frixx]: If $\Omega \subset \mathbb{C}$ is a open connected set then $\mathrm{H}(\Omega)$ is an EDD . (See Example 17.) As $\mathrm{H}(\Omega)$ is not UFD, it follows that $\mathrm{H}(\Omega)$ is not PID. (Contrary to [McD84, Exc. II.E.10 (b), p.144].)

Examples:

1. $\{1,-1\}$ is the set of units in $\mathbb{Z}$. A g.c.d. of $a_1,\ldots,a_k \in \mathbb{Z}$ is **uniquely normalized** by the condition $(a_1,\ldots,a_k) \geq 0$.

2. A positive integer $p \in \mathbb{Z}$ is irreducible if and only if p is prime.

3. $\mathbb{Z}_m$ is an integral domain, and hence a field, with m elements if and only if p is a prime.

4. Any nontrivial ideal in $\mathbb{Z}$ is $\mathbb{Z}_k$ for some positive integer k.

5. $\mathbb{Z} \supset I$ is a prime ideal if and only if all elements of I are divisible by some prime p.

6. $\{1,-1\}$ is the set of units in $\mathbb{Z}[x]$. A g.c.d. of $p_1,\ldots,p_k \in \mathbb{Z}[x]$, is **uniquely normalized** by the condition $(p_1,\ldots,p_k) = \sum_{i=0}^{m} a_i x^{m-i}$ and $a_0 \geq 0$.

7. Any prime element in $p(x) \in \mathbb{Z}[x], \deg p \geq 1$ is a primitive polynomial.

8. Let $p(x) = 2x+3, q(x) = 5x-3 \in \mathbb{Z}[x]$. Be two primitive polynomials. Clearly $(p(x),q(x)) = 1$. However 1 can not be expressed as $1 = a(x)p(x)+b(x)q(x)$, where $a(x),b(x) \in \mathbb{Z}[x]$. Indeed, if this was possible, then $1 = a(0)p(0)+b(0)q(0) = 3(a(0)-b(0))$, which is impossible for $a(0),b(0) \in \mathbb{Z}$. Hence $\mathbb{Z}[x]$ is not BD.

9. The field of quotients of $\mathbb{Z}$ is the field of rational numbers $\mathbb{Q}$.

10. Let $p(x),q(x) \in \mathbb{Z}[x]$ be two nonzero polynomials. Let $(p(x),q(x))$ be the g.c.d of p,q in $\mathbb{Z}[x]$. Use the fact that $p(x),q(x) \in \mathbb{Q}[x]$ to deduce that there exists a positive integer m and $a(x),b(x) \in \mathbb{Z}[x]$ such that $a(x)p(x)+b(x)q(x) = m(p(x),q(x))$. Furthermore, if $c(x)p(x)+d(x)q(x) = l(p(x),q(x))$ for some $c(x),d(x) \in \mathbb{Z}[x]$ and $0 \neq l \in \mathbb{Z}$ then $m|l$.

11. The set of real numbers $\mathbb{R}$ and the set of complex numbers $\mathbb{C}$ are fields.

12. A g.c.d. of $a_1,\ldots,a_k \in F[x]$ is **uniquely normalized** by the condition $(p_1,\ldots,p_k)$ is a monic polynomial.

13. A linear polynomial in $\mathbb{D}[\mathbf{x}]$ is irreducible.

14. Let $\Omega \subset \mathbb{C}$ be a connected set. Then each irreducible elements of $\mathrm{H}(\Omega)$ is an associate of $z-\zeta$ for some $\zeta \in \Omega$.

15. For $\zeta \in \mathbb{C}$ H_ζ every irreducible element is of the form $a(z-\zeta)$ for some $0 \neq a \in \mathbb{C}$. A g.c.d. of $a_1,\ldots,a_k \in \mathrm{H}_\zeta$ is **uniquely normalized** by the condition $(a_1,\ldots,a_k) = (z-\zeta)^m$ for some nonnegative integer m.

16. In $\mathrm{H}(\Omega)$, the set of functions which vanishes on a prescribed set $U \subseteq \Omega$, i.e.
$$I(U) := \{f \in \mathrm{H}(\Omega) : \quad f(\zeta) = 0,\ \zeta \in U\},$$
is an ideal.

17. Let Ω be an open connected set in $\mathbb{C}$. [Rud74, Theorem 15.11, 15.13] implies the following:

    - $I(U) \neq \{0\}$ if and only if U is a countable set, with no accumulations points in Ω.
    - Let U be a countable subset of Ω with no accumulation points in Ω. Assume that for each $\zeta \in U$ one is given a nonnegative integer $m(\zeta)$ and $m(\zeta)+1$ complex numbers $w_{0,\zeta}, ..., w_{m(\zeta),\zeta}$. Then there exists $f \in \mathrm{H}(\Omega)$ such that $f^{(n)}(\zeta) = n! w_{n,\zeta}, n = 0, ..., m(\zeta)$, for all $\zeta \in U$. Furthermore, if all $w_{n,\zeta} = 0$ then there exists $g \in \mathrm{H}(\Omega)$ such that all zeros of g are in U and g has a zero of order $m(\zeta)+1$ at each $\zeta \in U$.
    - Let $a, b \in \mathrm{H}(\Omega), ab \neq 0$. Then there exists $f \in \mathrm{H}(\Omega)$ such that $a = cf, b = df$. where $c, d \in \mathrm{H}(\Omega)$ and c, d do not have a common zero in Ω.
    - Let $c, d \in \mathrm{H}(\Omega)$ and assume that c, d do not have a common zero in Ω. Let U be the zero set of c in Ω, and denote by $m(\zeta) \geq 1$ the multiplicity of the zero $\zeta \in U$ of c. Then there exists $g \in \mathrm{H}(\Omega)$ such that $(e^g)^{(n)}(\zeta) = d^{(n)}(\zeta)$ for $n = 0, ..., m(\zeta)$, for all $\zeta \in U$. Hence $p = \frac{e^g - d}{c} \in \mathrm{H}(\Omega)$ and $e^g = pc + d$ is a unit in $\mathrm{H}(\Omega)$.
    - For $a, b \in \mathrm{H}(\Omega)$ there exists $p \in \mathrm{H}(\Omega)$ such $(a, b) = pa + b$.
    - For $a, b, c \in \mathrm{H}(\Omega)$ one has $(a, b, c) = p(a, b) + c = p(xa + b) + c$. Hence $\mathrm{H}(\Omega)$ is EDD.

18. Let $I \subset \mathbb{C}[x, y]$ be the ideal given by given by the condition $p(0, 0) = 0$. Then I is generated by x and y, and $(x, y) = 1$. I is not principal and $\mathbb{C}[x, y]$ is not BD.

19. $\mathbb{D}[x, y]$ is not BD.

3 Equivalence of Matrices

In this section we introduce matrices over an integral domain. Since any domain $\mathbb{D}$ can be viewed as a subset of its quotient field F, the notion of determinant, minor, rank and adjugate in Chapter 1 can be applied to these matrices. It is an interesting problem to determine whether one given matrix can be transformed to another by left multiplication, right multiplication, or multiplication on both sides, using only matrices invertible with the domain.

These are equivalence relations and the problem is to characterize left (row) equivalence classes, right (columns) equivalence classes and equivalence classes in $\mathbb{D}^{m\times n}$. For BD the left equivalence classes are characterized by their Hermite normal form, which attributed to Hermite. For EDD the equivalence classes are characterized by their Smith normal form [Smi61]. Most of the results of this section can be found in [McD84]. Some special results of this section are given in [Fri81] and [Frixx].

Definitions:
For a set S denote by $S^{m\times n}$ the set of all $m \times n$ matrices $A = [a_{ij}]_{i=j=1}^{i=m,j=n}$, where each $a_{ij} \in S$.
For positive integers $p \leq q$ denote by $Q_{p,q}$ is the set of all subsets $\{i_1, \ldots, i_p\} \subset \{1, 2, \ldots q\}$ of cardinality p, where we assume that $1 \leq i_1 < \ldots < i_p \leq q$. $\binom{q}{p}$ is the cardinality of $Q_{p,q}$.
null $A := n -$ rank A is the **nullity** of $A \in \mathbb{D}^{m\times n}$.
$U \in \mathbb{D}^{n\times n}$ is called **invertible**, (**unimodular**), if there exists $V \in \mathbb{D}^{n\times n}$ such that $UV = VU = I_n$. So $1 = \det U \det V$ and $\det U$ is a unit in $\mathbb{D}$. Vice versa if $\det U$ is a unit then U is invertible, and its inverse U^{-1} is given by $U^{-1} = \det{}^{-1} U \text{adj } U$.
$\mathbf{GL}(n, \mathbb{D})$ denotes the group of invertible matrices in $\mathbb{D}^{n\times n}$.
Let $A, B \in \mathbb{D}^{m\times n}$. Then A and B are column equivalent, row equivalent and equivalent if the following conditions hold respectively:

$$
\begin{aligned}
&B = AP \quad \text{for some } P \in \mathbf{GL}(n, \mathbb{D}) \quad (A \sim_c B),\\
&B = QA \quad \text{for some } Q \in \mathbf{GL}(m, \mathbb{D}) \quad (A \sim_r B),\\
&B = QAP \quad \text{for some } P \in \mathbf{GL}(n, \mathbb{D}),\ Q \in \mathbf{GL}(m, \mathbb{D}) \quad (A \sim B).
\end{aligned}
$$

For $A \in \mathbb{D}_g^{m\times n}$ let

$$
\begin{aligned}
&\mu(\alpha, A) := \text{g.c.d. } \{\det A[\alpha, \theta],\ \theta \in Q_{k,n}\}, \quad \alpha \in Q_{k,m},\\
&\nu(\beta, A) := \text{g.c.d. } \{\det A[\phi, \beta],\ \phi \in Q_{k,m}\}, \quad \beta \in Q_{k,n},\\
&\delta_k(A) := \text{g.c.d. } \{\det A[\phi, \theta],\ \phi \in Q_{k,m},\ \theta \in Q_{k,n}\},
\end{aligned}
$$

$\delta_k(A)$ is called the k-th **determinant invariant** of A.
For $A \in \mathbb{D}_g^{m\times n}$

$$
\begin{aligned}
&i_j(A) := \frac{\delta_j(A)}{\delta_{j-1}(A)}, \quad j = 1, ..., \text{rank } A, \quad (\delta_0(A) = 1),\\
&i_j(A) = 0 \quad \text{for rank } A < j \leq \min(m, n),
\end{aligned}
$$

are called the **invariant factors** of A. $i_j(A)$ is called a **trivial** factor if $i_j(A)$ is unit in $\mathbb{D}_g$. We adopt the **normalization** $i_j(A) = 1$ for any trivial factor of A. For $\mathbb{D} = \mathbb{Z}, \mathbb{Z}[x], F[x]$ we adopt the normalizations given in previous section in the Examples 1, 6, 12 respectively.

Assume that $\mathbb{D}[x]$ is GCDD. Then the invariant factors of $A \in \mathbb{D}[x]^{m\times n}$ are also called **invariant polynomials**.
$D = (d_{ij}) \in \mathbb{D}^{m\times n}$ is called a **diagonal** matrix if $d_{ij} = 0$ for all $i \neq j$. The entries $d_{11}, ..., d_{\ell\ell}$, $\ell = \min(m, n)$ are called the diagonal entries of D. D is denoted as $D = \text{diag}\,(d_{11}, ..., d_{\ell\ell}) \in \mathbb{D}^{m\times n}$.
Denote by $\Pi_n \subset \mathbf{GL}(n, \mathbb{D})$ the group of $n \times n$ permutation matrices.
An invertible matrix $U \in \mathbf{GL}(n, \mathbb{D})$ is called **simple** if there exists $P, Q \in \Pi_n$ such that $U = P \begin{bmatrix} V & 0 \\ 0 & I_{n-2} \end{bmatrix} Q$, where $V = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \in \mathbf{GL}(2, \mathbb{D})$, i.e. $\alpha\delta - \beta\gamma$ is invertible.
U is called **elementary** if U is of the above form and $V = \begin{bmatrix} \alpha & 0 \\ \gamma & \delta \end{bmatrix} \in \mathbf{GL}(2, \mathbb{D})$, i.e. α, δ are invertible.
For $A \in \mathbb{D}^{m\times n}$ the following row (column) operations are called **elementary**:
(a) interchange any two rows (columns) of A;
(b) multiply row (column) i by an invertible element a;
(c) add to row (column) j b times row (column) i ($i \neq j$).
For $A \in \mathbb{D}^{m\times n}$ the following row (column) operations are called **simple**:
(d) replace row (column) i by a times row (column) i plus b times row (column) j, and row (column) j by c times row (column) i plus d times row (column) j, where $i \neq j$ and $ad - bc$ is invertible in $\mathbb{D}$.
$B = (b_{ij}) \in \mathbb{D}^{m\times n}$, is called to be in **Hermite normal** form if the following conditions hold:
Let $r = \text{rank}\ B$. First, $i - th$ row of B is a nonzero row if and only if $i \leq r$. Second, let b_{in_i} be the first nonzero entry in i-th row for $i = 1, ..., r$. Then $1 \leq n_1 < n_2 < \cdots < n_r \leq n$.
$B \in \mathbb{D}_g^{m\times n}$ is called to be in **Smith normal** form if B is a diagonal matrix $B = \text{diag}\,(b_1, ..., b_r, 0, ..., 0)$, $b_i \neq 0$ for $i = 1, \ldots, r$ and $b_{i-1}|b_i$ for $i = 2, \ldots, r$.
For a monic polynomial $p(x) = x^m + a_1x^{m-1} + ... + a_m \in \mathbb{D}[x]$

$$C(p) = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ldots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 0 & 1 \\ -a_m & -a_{m-1} & -a_{m-2} & \ldots & -a_2 & -a_1 \end{bmatrix} \in \mathbb{D}^{m\times m}$$

is called the **companion** matrix corresponding to p. Denote $C(p)(x) = -C(p) + xI$.

Facts:

1. For $A \in \mathbb{D}^{m\times n}$ the rank of A is the maximal size of the nonvanishing minor. (The rank of zero matrix is 0.)

2. Column equivalence, row equivalence and equivalence of matrices are equivalence relations in $\mathbb{D}^{m\times n}$.

3. For any $A, B \in \mathbb{D}^{m\times n}$ one has $A \sim_r B \iff A^T \sim_c B^T$. Hence it is enough to consider the row equivalence relation.

4. For $A, B \in \mathbb{D}_g^{m\times n}$ the Cauchy-Binet formula, (§1.4), yields

$$\begin{aligned}
&\mu(\alpha, A) \equiv \mu(\alpha, B) \quad \text{for all } \alpha \in Q_{k,m} \quad \text{if } A \sim_c B,\\
&\nu(\beta, A) \equiv \nu(\beta, B) \quad \text{for all } \beta \in \mathrm{Q}_{k,n} \quad \text{if } A \sim_r B,\\
&\delta_k(A) \equiv \delta_k(B) \quad \text{if } A \sim B,
\end{aligned}$$

   for $k = 1, ..., \min(m, n)$.

5. The elementary row and column operations can be carried out by multiplications by A by suitable elementary matrices from the left and the right respectively.

6. The simple row and column operations are carried out by multiplications by A by suitable simple matrices U from the left and right respectively.

7. Let $\mathbb{D}_b$ be a Bezout domain, $A \in \mathbb{D}_b^{m\times n}$, rank $A = r$. Then A is row equivalent to $B = (b_{ij}) \in \mathbb{D}_b^{m\times n}$, in a Hermite normal form, which satisfies the following conditions.

   Let b_{in_i} be the first nonzero entry in i-th row for $i = 1, ..., r$. Then $1 \le n_1 < n_2 < \cdots < n_r \le n$ are uniquely determined and the elements $b_{in_i},\ i = 1, ..., r$ are uniquely determined, up to units, by the conditions

$$\begin{aligned}
&\nu((n_1, ..., n_i), A) = b_{1n_1} \cdots b_{in_i}, \quad i = 1, ..., r,\\
&\nu(\alpha, A) = 0,\ \alpha \in Q_{i,n_i-1}, \quad i = 1, ..., r.
\end{aligned}$$

   The elements $b_{jn_i},\ j = 1, ..., i-1$ are then successively uniquely determined up to the addition of arbitrary multiples of b_{in_i}. The remaining elements b_{ik} are now uniquely determined. The invertible matrix Q, such that $B = QA$, can be given by a finite product of simple matrices.

   If b_{in_i} in the Hermite normal form is invertible we assume the normalization conditions $b_{in_i} = 1$ and $b_{jn_i} = 0$ for $i < j$.

8. For Euclidean domains we assume normalization conditions either $b_{jn_i} = 0$ or $d(b_{jn_i}) < d(b_{in_i})$ for $j < i$. Then for any $A \in \mathbb{D}_e^{m\times n}$, in a Hermite normal form $B = QA,\ Q \in \mathbf{GL}_m(\mathbb{D}_e)$ Q is a product of a finite elementary matrices.

9. $U \in \mathbf{GL}(n, \mathbb{D}_e)$ is a finite product of elementary invertible matrices.

10. For $\mathbb{Z}$ we assume the normalization $b_{in_i} \ge 1$ and $0 \le b_{jn_i} < b_{in_i}$ for $j < i$. For $F[x]$ we assume that b_{in_i} is a monic polynomial and $\deg b_{jn_i} < \deg b_{in_i}$ for $j < i$. Then for $\mathbb{D}_e = \mathbb{Z}, F[x]$ any $A \in \mathbb{D}_e^{m\times n}$ has a unique Hermite normal form.

11. $A, B \in \mathbb{D}_b$ are row equivalent if and only if A and B are equivalent to the same Hermite canonical form.

12. $A \in F^{m\times n}$ can be brought to its unique Hermite normal form, called the *reduced row echelon form* (RREF),

$$b_{in_i} = 1,\ b_{jn_i} = 0, \quad j = 1, ..., i-1,\ i = 1, ..., r = \text{rank } A,$$

by a finite number of elementary row operations. Hence $A, B \in F^{m\times n}$ are row equivalent if and only $r = \text{rank } A = \text{rank } B$ they have the same RREF. (See §1.1.)

13. For $A \in \mathbb{D}_g^{m\times n}$ and $1 \le p < q \le \min(m, n)$ $\delta_p(A)|\delta_q(A)$.

14. For $A \in \mathbb{D}_g^{m\times n}$ $i_{j-1}(A)|i_j(A)$ for $j = 2, ..., \text{rank } A$.

15. Any $0 \ne A \in \mathbb{D}_{ed}^{m\times n}$ is equivalent to its Smith normal form $B = \text{diag}\,(i_1(A), ..., i_r(A), 0, ..., 0)$, where $r = \text{rank } A$ and $i_1(A), \ldots, i_r(A)$ are the invariants factors of A.

16. $A, B \in \mathbb{D}_{ed}^{m\times n}$ are equivalent if and only if A and B have the same rank rank and the same invariant factors.

Examples:

1. Let $A = \begin{bmatrix} 1 & a \\ 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 1 & b \\ 0 & 0 \end{bmatrix} \in \mathbb{D}^{2\times 2}$ be two Hermite normal forms. It is straightforward to show that $A \sim_r B$ if and only if $a = b$. Assume that $\mathbb{D}$ is BD and let $a \ne 0$. Then rank $A = 1, \nu((1), A) = 1, \{\{\nu((2), A)\}\} = \{\{a\}\}, \nu((1, 2), A) = 0$. If $\mathbb{D}$ has other units then 1, it follows that $\nu(\beta, A)$ for all $\beta \in Q_{k,2}, k = 1, 2$ do not determine the row equivalence class of A.

2. Let $A = \begin{bmatrix} a & c \\ b & d \end{bmatrix} \in \mathbb{D}_b^{2\times 2}$. Then there exists $u, v \in \mathbb{D}_b$ such that $ua + vb = (a, b) = \nu((1), A)$. If $(a, b) \ne 0$ then $1 = (u, v)$. If $a = b = 0$ choose $u = 1, v = 0$. Hence there exists $x, y \in \mathbb{D}_b$ such that $yu - xv = 1$. Thus $V = \begin{bmatrix} u & v \\ x & y \end{bmatrix} \in \mathbf{GL}(2, \mathbb{D}_b)$ and $VA = \begin{bmatrix} (a, b) & c' \\ b' & d' \end{bmatrix}$. Clearly $b' = xa + yb = (a, b)e$. Hence $\begin{bmatrix} 1 & 0 \\ -e & 1 \end{bmatrix} VA = \begin{bmatrix} (a, b) & c' \\ 0 & f \end{bmatrix}$ is a Hermite normal form of A. This construction is easily extended to obtain a Hermite normal form for any $A \in \mathbb{D}_b^{m\times n}$, using simple row operations.

3. Let $A \in \mathbb{D}_{ed}^{2\times 2}$ as in the previous example. Assume that $ab \ne 0$. Change the two rows of A if needed to assume that $d(a) \le d(b)$. Let $a_1 = b, a_2 = a$ and do the first step of Euclid's Algorithm: $a_1 = t_1 a_2 + a_3$, where $a_3 = 0$ or $d(a_3) < d(a_2)$. Let $V = \begin{bmatrix} 1 & 0 \\ -t_1 & 1 \end{bmatrix} \in \mathbf{GL}(2, \mathbb{D}_{ed})$ be

an elementary matrix. Then $A_1 = VA = \begin{bmatrix} a_2 & * \\ a_3 & * \end{bmatrix}$. If $a_3 = 0$ then A_1 has a Hermite normal form. If $a_3 \neq 0$ continue as above. Since Euclid's Algorithm terminates after a finite number of steps it follows that A can be put into Hermite normal form by a finite number of elementary row operations. This statement holds similarly in the case $ab = 0$. This construction is easily extended to obtain a Hermite normal form for any $A \in \mathbb{D}_b^{m\times n}$, using elementary row operations.

4. Assume that D is BD and let $A = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \in \mathbb{D}^{2\times 2}$. Note that $\delta_1(A) = (a, b, c)$. If A is equivalent to a Smith normal form then there exists $V, U \in \mathbf{GL}(2, D)$ such that $VAU = \begin{bmatrix} (a, b, c) & * \\ * & * \end{bmatrix}$. Without a loss of generality we may assume that $\det V = \det U = 1$. Assume that $V = \begin{bmatrix} p & q \\ \overline{q} & \overline{p} \end{bmatrix}, U = \begin{bmatrix} x & \overline{y} \\ y & \overline{x} \end{bmatrix}$. Then this condition is equivalent to the existence $p, q, x, y \in \mathbb{D}$ such that $(px)a + (py)b + (qy)c = (a, b, c)$. Thus, if each $A \in \mathbb{D}^{2\times 2}$ is equivalent to Smith normal form $\mathbb{D}$ it follows that $\mathbb{D}$ is EDD.

   Vice versa suppose that $\mathbb{D}$ is EDD. Then $\mathbb{D}$ is BD. Let $A \in \mathbb{D}^{2\times 2}$. First, bring A to an upper triangular Hermite normal form using simple row operations: $A_1 = WA = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix}, W \in \mathbf{GL}(2, \mathbb{D}_b)$. Note that $\delta_1(A) = \delta_1(A_1) = (a, b, c)$. Since $\mathbb{D}$ is EDD there exists $p, q, x, y \in \mathbb{D}$ such that $(px)a+(py)b+(qy)c = (a, b, c)$. If $(a, b, c) \neq (0, 0, 0)$ then $(p, q) = (x, y) = 1$. Otherwise $A = A_1 = 0$ and we are done. Hence there exist $\overline{p}, \overline{q}, \overline{x}, \overline{y}$ such that $p\overline{p} - q\overline{q} = x\overline{x} - y\overline{y} = 1$. Let $V = \begin{bmatrix} p & q \\ \overline{q} & \overline{p} \end{bmatrix}, U = \begin{bmatrix} x & \overline{y} \\ y & \overline{x} \end{bmatrix}$. Thus $G = VA_1U = \begin{bmatrix} \delta_1(A) & g_{12} \\ g_{21} & g_{22} \end{bmatrix}$. Since $\delta_1(G) \equiv \delta_1(A)$ we deduce that $\delta_1(A)$ divides g_{12} and g_{21}. Apply appropriate elementary row and column operations to deduce that A is equivalent to a diagonal matrix $C = \operatorname{diag}(i_1(A), d_2)$. As $\delta_2(C) = i_1(A)d_2 \equiv \delta_2(A)$ we see that C is has Smith normal form.

   These arguments are easily extended to obtain a Smith normal form for any $A \in \mathbb{D}_{ed}^{m\times n}$, using simple row and column operations.

5. Assume that $\mathbb{D}$ is ED. Let $0 \neq A \in \mathbb{D}^{2\times 2}$. Permute the rows and the columns of A to obtain $A_1 = [a_{ij,1}]$ such that $a_{11,1} \neq 0$ and $d(a_{11,1}) = \min_{i,j\in[1,2]} d(a_{ij,1})$. First use elementary row operations to bring A_1 to Hermite normal form $A_2 = [a_{ij,2}]$. And then use elementary column operations to bring to obtain $A_3 = [a_{ij,3}]$, such that A_3^{T} is a Hermite normal form. Clearly $d(a_{11,3}) \leq d(a_{11,1})$. If equality holds then we may assume that A_3 is a diagonal matrix. If A_3 is not a diagonal matrix we

know that $d(a_{11,3}) < d(a_{11,1})$. Now apply the above process to A_3 and so on. Hence, there exists $k \geq 3$ such that A_k must be diagonal. Add the second row to the first column to get A_{k+1}. Continue in this manner until one obtains a diagonal matrix A_l in the Smith normal form.

These arguments are easily extended to obtain a Smith normal form for any $A \in \mathbb{D}_e^{m \times n}$, using elementary row and column operations.

6. Let $\mathbb{D}$ be an integral domain and assume that Let $p(x) = x^m + \sum_{i=1}^{m} a_i x^{m-i} \in \mathbb{D}[x]$ is a monic polynomial of degree $m \geq 2$. Let $C(p) \in \mathbb{D}^{m \times m}$ be the companion matrix. Then $\det(xI_m - C(p)) = p(x)$. Assume that $\mathbb{D}[x]$ is GCDD. Recall that $C(p)(x) = xI_m - C(p) \in \mathbb{D}[x]^{n \times n}$. Since $-I_{m-1}$ is a submatrix of $C(p)(x)$, it follows that $\delta_i(C(p)(x)) = 1$ for $i = 1, \ldots, m-1$. Hence the invariant factors of $C(p)(x)$ are $i_1(C(p)(x)) = \ldots = i_{m-1}(C(p)(x)) = 1$ and $i_m(C(p)(x)) = p(x)$. If $\mathbb{D}$ is a field the $C(p)(x)$ is equivalent over $\mathbb{D}[x]$ to $\operatorname{diag}(1, \ldots, 1, p(x))$.

7. Let $p_1(x), \ldots, p_k(x) \in \mathbb{D}[x]$ be $k \geq 2$ monic polynomials, where $\deg p_1 \geq 1$ and $p_i | p_{i+1}$ for $i = 1, \ldots, k-1$. Let $n = \sum_{i=1}^{k} \deg p_i$ and $A = \operatorname{diag}(C(p_1), \ldots, C(p_k)) \in \mathbb{D}^{n \times n}$. Assume that $\mathbb{D}[x]$ is GCDD. Then the invariant factors of $A(x) = xI_n - A$ are $i_j(A(x)) = 1$ for $j = 1, \ldots, n-k$ and $i_{n-k+j}(A(x)) = p_j(x)$ for $j = 1, \ldots, k$.

4 Linear equations over Bezout domains

The standard notions of modules can be found in [ZS58] and [McD84]. The solvability of linear systems over EDD can be traced to in [Hel43] and [Kap49]. The results for BD can be found in [Fri81] and [Frixx]. The results for H_0 are given in [Fri80]. The general theory of solvability of the systems of equations over commutative rings is discussed in [McD84, Exc. I.G.7-I.G.8].

Definitions:
$\mathbf{M}$ is called a $\mathbb{D}$-**module**, if $\mathbf{M}$ is an additive group with respect to + operations, and $\mathbf{M}$ admits a multiplications by a *scalar* $a \in \mathbb{D}$, i.e. there exists a mapping $\mathbb{D} \times \mathbf{M} \to \mathbf{M}$ which satisfies the the standard distribution properties. (For a field $\mathbb{F}$ $\mathbf{M}$ is a vector space over $\mathbb{F}$.)
$\mathbf{M}$ is **finitely generated** if there exists $\mathbf{v}_1, \ldots, \mathbf{v}_n$ so that every $\mathbf{v} \in \mathbf{M}$ is a linear combination of $\mathbf{v}_1, \ldots, \mathbf{v}_n$, over $\mathbb{D}$, i.e. $\mathbf{v} = a_1\mathbf{v}_1 + \ldots + a_n\mathbf{v}_n$ for some $a_1, \ldots, a_n \in \mathbb{D}$.
$\mathbf{v}_1, \ldots, \mathbf{v}_n$ is called a **basis** of $\mathbf{M}$, every $\mathbf{v}$ can be written as a unique linear combination of $\mathbf{v}_1, \ldots, \mathbf{v}_n$.
$\dim \mathbf{M} = n$ means that $\mathbf{M}$ has a basis of n elements.
$\mathbf{N} \subset \mathbf{M}$ is called a $\mathbb{D}$-**submodule** of $\mathbf{M}$, if $\mathbf{N}$ is closed under the addition and multiplication by scalars.
$\mathbb{D}^n := \mathbb{D}^{n \times 1}$ is a $\mathbb{D}$-module. It has a **standard** basis $\mathbf{e}_i$ for $i = 1, \ldots, n$, where $\mathbf{e}_i$ is the $i-th$ column of the identity matrix I_n.

For any $A \in \mathbb{D}^{m\times n}$ the **range** of A, denoted by range(A), is the set of all linear combinations of the columns of A.
The **kernel** of A, denoted by ker(A), is the set of all solutions to the homogeneous equation $A\mathbf{x} = \mathbf{0}$.
Consider a system of m linear equations in n unknowns: $\sum_{j=1}^{n} a_{ij}x_j = b_j, \quad i = 1, ..., m$, where $a_{ij}, b_i \in \mathbb{D}$ for $i = 1, ..., m,\ j = 1, ..., n$. In matrix notation this system is $A\mathbf{x} = \mathbf{b}$,where $A = [a_{ij}] \in \mathbb{D}^{m\times n},\ \mathbf{x} := [x_1, \ldots, x_n]^T \in \mathbb{D}^n,\ \mathbf{b} := [b_1, \ldots, b_m]^T \in \mathbb{D}^m$. A and $[A, \mathbf{b}] \in \mathbb{D}^{m\times(n+1)}$ are called the **coefficient** matrix and the **augmented** matrix respectively.
Let $A \in \mathrm{H}_0^{m\times n}$. Then $A = A(z) = (a_{ij}(z))_{i=j=1}^{m,n}$ and $A(z)$ has the McLaurin expansion $A(z) = \sum_{k=0}^{\infty} A_k z^k$, where $A_k \in \mathbb{C}^{m\times n}, k = 0, ...$ Here each $a_{ij}(z)$ has convergent McLaurin series for $|z| < R(A)$ for some $R(A) > 0$.
The invariant factors of A are called the **local invariant polynomials** of A, which are normalized to be of the form $i_k(A) = z^{i_k}$ for $0 \le i_1 \le i_2 \le ... \le i_r$, where $r = \text{rank } A$.
The integer i_r is called the **index** of A and is denoted by $\eta = \eta(A)$. For a nonnegative integer p denote by $\kappa_p = \kappa_p(A)$-the number of local invariant polynomials of A whose degree is equal to p.

Facts:

1. The system $A\mathbf{x} = \mathbf{b}$ is solvable over a Bezout domain $\mathbb{D}_b$ if and only if $r = \text{rank } A = \text{rank } [A, \mathbf{b}]$ and $\delta_r(A) \equiv \delta_r([A, \mathbf{b}])$, which is equivalent to the statement that A and $[A, \mathbf{b}]$ have the same set invariant polynomials, up to invertible elements.

   For a field $\mathbb{F}$ this result reduces to the equality rank $A = \text{rank } [A, \mathbf{b}]$.

2. For $A \in \mathbb{D}_b^{m\times n}$ range$A$ and ker A are modules in $\mathbb{D}_b^m$ and $\mathbb{D}_b^n$ having finite bases with rank A and null A elements respectively. Moreover the basis of ker A can be completed to a basis of $\mathbb{D}_b^n$.

3. For $A, B \in \mathrm{H}_0^{m\times n}$ let $C(z) = A(z) + z^{k+1}B(z)$, where k is a nonnegative integer. Then A and C have the same local invariant polynomials up to degree k. Moreover, if k is equal to the index of A, and A and C have the same rank then A is equivalent to C.

4. Consider a system of linear equations over H_0 $A(z)\mathbf{u}(z) = \mathbf{b}(z)$, where $A(z) \in \mathrm{H}_0^{m\times n}$ and $\mathbf{b}(z) = \sum_{k=0}^{\infty} \mathbf{b}_k z^k \in \mathrm{H}_0^m, \quad \mathbf{b}_k \in \mathbb{C}^m,\ k = 0, ...$ Look for the power series solution $\mathbf{u}(z) = \sum_{k=0}^{\infty} \mathbf{u}_k z^k$, where $\mathbf{u}_k \in \mathbb{C}^n,\ k = 0, ...,$. Then $\sum_{j=0}^{k} A_{k-j}\mathbf{u}_j = \mathbf{b}_k$, for $k = 0, ...$. This system is solvable for $k = 0, ..., q \in \mathbb{Z}_+$ if and only if $A(z)$ and $[A(z), \mathbf{b}(z)]$ have the same local invariant polynomials up to degree q.

5. Suppose that $A(z)\mathbf{u}(z) = \mathbf{b}(z)$ is solvable over H_0. Let $q = \eta(A)$ and suppose that $\mathbf{u}_0, ..., \mathbf{u}_q$ satisfies the system of equations, given in the

previous fact, for $k = 0, ..., q$. Then there exists a solution $\mathbf{u}(z) \in \mathrm{H}_0^n$ satisfying $\mathbf{u}(0) = \mathbf{u}_0$.

6. Let $q \in \mathbb{Z}_+$ and $\mathbf{W}_q \subset \mathbb{C}^n$ be the subspace of all vectors $\mathbf{w}_0$ such that $\mathbf{w}_0, ..., \mathbf{w}_q$ is a solution to the homogenous system $\sum_{j=0}^{k} A_{k-j}\mathbf{w}_j = 0$, for $k = 0, ..., q$. Then $\dim \mathbf{W}_q = n - \sum_{j=0}^{q} \kappa_j(A)$. In particular, for $\eta = \eta(A)$ and any $\mathbf{w}_0 \in \mathbf{W}_\eta$ there exists $\mathbf{w}(z) \in \mathrm{H}_0^n$ such that $A(z)\mathbf{w}(z) = \mathbf{0}, \quad \mathbf{w}(0) = \mathbf{w}_0$.

5 Strict equivalence of pencils

The notion of strict equivalence of $n \times n$ regular pencils over the fields goes back to K. Weierstrass [Wei67]. The notion of strict similarity of $m \times n$ matrices over the fields is due to L. Kronecker [Kro90]. Most of the details can be found in [Gan59]. Some special results are proven in [Frixx].

Definitions:
A matrix $A(x) \in \mathbb{D}[x]^{m\times n}$ is called a **pencil** if $A(x) = A_0 + xA_1,\ A_0, A_1 \in \mathbb{D}^{m\times n}$.
A pencil $A(x)$ is called **regular** if $m = n$ and $\det A(x)$ is not the zero polynomial. Otherwise $A(x)$ is called a **singular** pencil.
Associate with a pencil $A(x) = A_0 + xA_1 \in \mathbb{D}[x]$ the **homogeneous** pencil $A(x_0, x_1) = x_0A_0 + x_1A_1 \in \mathbb{D}[x_0, x_1]$.
Two pencils $A(x), B(x) \in \mathbb{D}[x]^{m\times n}$ are called **strictly** equivalent, denoted by $A(x) \overset{s}{\sim} B(x)$, if $B(x) = QA(x)P$ for some $P \in \mathbf{GL}_n(\mathbb{D}),\ Q \in \mathbf{GL}_m(\mathbb{D})$.
Similarly, two homogeneous pencils $A(x_0, x_1), B(x_0, x_1) \in \mathbb{D}[x_0, x_1]^{m\times n}$ are called **strictly** equivalent, denoted by $A(x_0, x_1) \overset{s}{\sim} B(x_0, x_1)$, if $B(x_0, x_1) = QA(x_0, x_1)P$ for some $P \in \mathbf{GL}_n(\mathbb{D}),\ Q \in \mathbf{GL}_m(\mathbb{D})$.
For UFD $\mathbb{D}$ let $\delta_k(x_0, x_1),\ i_k(x_0, x_1)$ be the invariant determinants and factors of $A(x_0, x_1)$ respectively for $k = 1, ..., \text{rank } A(x_0, x_1)$. They are called **homogeneous** determinants and the invariant **homogeneous** polynomials (factors) respectively of $A(x_0, x_1)$. (Sometimes, $\delta_k(x_0, x_1),\ i_k(x_0, x_1), k = 1, ..., \text{rank } A(x_0, x_1)$ are called the homogeneous determinants and the invariant homogeneous polynomials $A(x)$.)
Let $A(x) \in \mathbb{F}[x]^{m\times n}$ and consider the module $\mathbf{M} \subset \mathbb{F}[x]^n$ of all solutions of $A(x)\mathbf{w}(x) = 0$. The set of all solutions $\mathbf{w}(x)$ is an $\mathbb{F}[x]$-module $\mathbf{M}$ with a finite basis $\mathbf{w}_1(x), ..., \mathbf{w}_s(x)$, where $s = n - \text{rank } A(x)$. Choose a basis $\mathbf{w}_1(x), ..., \mathbf{w}_s(x)$ in $\mathbf{M}$ such that $\mathbf{w}_k(x) \in \mathbb{M}$ has the lowest degree among all $\mathbf{w}(x) \in \mathbf{M}$ which are linearly independent over $\mathbb{F}(x)$ of $\mathbf{w}_1, ..., \mathbf{w}_{k-1}(x)$ for $k = 1, ..., s$. Then the **column indices** $\alpha_1 \le \alpha_2 \le ... \le \alpha_s$ of $A(x)$ are given as $\alpha_k = \deg \mathbf{w}_k(x)$, $k = 1, ..., s$. The **row indices** $0 \le \beta_1 \le \beta_2 \le ... \le \beta_t,\ t = m - \text{rank } A(x)$, of $A(x)$ are the column indices of $A(x)^{\mathrm{T}}$.

Facts:

1. Let $A_0, A_1, B_0, B_1 \in \mathbb{D}^{m\times n}$. Then $A_0 + xA_1 \overset{s}{\sim} B_0 + xB_0 \iff x_0A_0 + x_1A_1 \overset{s}{\sim} B_0x_0 + B_1x_1$.

2. Let $A_0, A_1 \in \mathbb{D}_u$. Then the invariant determinants and the invariant polynomials $\delta_k(x_0, x_1)$, $i_k(x_0, x_1)$, $k = 1, ..., \text{rank } x_0A_0 + x_1A_1$, of $x_0A_0 + x_1A_1$ are homogeneous polynomials. Moreover, if $\delta_k(x)$ and $i_k(x)$ are the invariant determinants and factors of the pencil $A_0 + xA_1$ for $k = 1, ..., \text{rank } A_0 + xA_1$, then $\delta_k(x) = \delta_k(1, x)$, $i_k(x) = i_k(1, x)$, for $k = 1, ..., \text{rank } A_0 + xA_1$.

3. [Wei67]: Let $A_0 + xA_1 \in \mathbb{F}[x]^{n\times n}$ be a regular pencil. Then a pencil $B_0 + xB_1 \in \mathbb{F}[x]^{n\times n}$ is strictly equivalent to $A_0 + xA_1$ if and only if $A_0 + xA_1$ and $B_0 + xB_1$ have the same invariant polynomials over $\mathbb{F}[x]$.

4. [Frixx]: Let $A_0 + xA_1, B_0 + xB_1 \in \mathbb{D}[x]^{n\times n}$. Assume that $A_1, B_1 \in \mathbf{GL}_n(\mathbb{D})$. Then $A_0 + xA_1 \overset{s}{\sim} B_0 + xB_1 \iff A_0 + xA_1 \sim B_0 + xB_1$.

5. [Gan59]: The column (row) indices are independent of a particular allowed choice of a basis $\mathbf{w}_1(x), ..., \mathbf{w}_s(x)$.

6. For singular pencils the invariant homogeneous polynomials alone do not determine the class of strictly equivalent pencils.

7. [Kro90, Gan59]: The pencils $A(x), B(x) \in \mathbb{F}[x]^{m\times n}$ are strictly equivalent if and only if they have the same invariant homogeneous polynomials and the same row and column indices.

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It turns out that linear transformations can be represented in a 1-1 fashion in matrices. This chapter will be most likely be a review as the topic has already probably been covered in high school (see this link). The establishment of a one-to-one correspondence between linear transformations and matrices is very important in the study of linear transformations. Suppose you have a set of basis vectors x1, x2, x3, ..., xm of a vector space X and basis vectors y1, y2, y3, ..., yn of a vector space Y The Linear Algebra topics include matrix operations, determinants and systems of linear equations. In the section â€œVector Algebraâ€, a main attention is paid to the geometrical applications of vector operations. The vector approach is considered to be basic for discussion of classic problems of Analytical Geometry. The author welcomes readerâ€™s suggestions for improvement of future editions of this textbook.Â Matrices allow us to operate with arrays consisting of many numbers, functions or mathematical statements, just as if we operate with several items. Matrices have a wide application in different branches of knowledge, for instance, in mathematics, physics, computer science, and so on. In linear algebra, functions will again be the focus of your attention, but functions of a very special type. In precalculus you were perhaps encouraged to think of a function as a machine f into which one may feed a real number. For each input x this machine outputs a single real number f (x). 15.Â While this sounds complicated, linear algebra is the study of simple func-tions of vectors; its time to describe the essential characteristics of linear functions. Letâ€™s use the letter L to denote an arbitrary linear function and think again about vector addition and scalar multiplication. Also, suppose that v and u are vectors and c is a number.Â 6The domain, codomain, and rule of correspondence of the function are represented by the left blog, right blob, and arrows, respectively. 17. Matrix functions are used in many areas of linear algebra and arise in numerous applica-tions in science and engineering. The most common matrix function is the matrix inverse; it is not treated specically in this article, but is covered in Chapter B. This article is concerned with general matrix functions as well as the specic cases of matrix square roots, trigonometric functions, and the exponential and logarithmic functions.Â For example, cos(A) = I â^' A2/2! + A4/4! â^' Â· Â· Â·. However, a general theory exists from which a number of properties possessed by all matrix functions can be deduced and which suggest computational methods. This article treats general theory, then specic functions, and nally outlines computational methods.