

## A GLOBAL EXISTENCE AND UNIQUENESS RESULT FOR FUNCTIONAL INTEGRO-DIFFERENTIAL EQUATIONS IN BANACH SPACES

BY

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**Abstract.** We provide in this paper, sufficient conditions for the existence of the unique mild solution on the semi-infinite interval for fractional integro-differential equations with time delay by using a recent nonlinear alternative of Leray-Schauder type due to Frigon and Granas for contractions maps in Fréchet spaces.

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**Key words:** integral resolvent family, mild solution, fixed points, Fréchet spaces, admissible contractions.

### 1. Introduction

In this paper, we study the existence of mild solutions, defined on the positive semi-infinite real interval  $J := [0, +\infty)$ , for semilinear integro-differential equations of fractional order

$$(1) \quad y'(t) - \int_0^t \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)} Ay(s)ds = f(t, y_t), \quad \text{a.e. } t \in J,$$

$$(2) \quad y_0 = \phi \in \mathcal{B},$$

where  $1 < \alpha < 2$  and  $A : D(A) \subset E \rightarrow E$  is the generator of an integral resolvent family defined on a complex Banach space  $(E, |\cdot|)$ , the convolution integral in the equation is known as the Riemann-Liouville fractional integral and  $f : [0, +\infty) \times E \rightarrow E$  is a given function. For any continuous function  $y$  defined on  $(-\infty, +\infty)$  and any  $t \geq 0$ , we denote by  $y_t$  the element of  $\mathcal{B}$  defined by  $y_t(\theta) = y(t + \theta)$  for  $\theta \in (-\infty, 0]$ . Here  $y_t(\cdot)$  represents the

history of the state from each time  $\theta \in (-\infty, 0]$  up to the present time  $t$ . We assume that the histories  $y_t$  belongs to some abstract phase space  $\mathcal{B}$ , to be specified later.

Integro-differential equations arise in many engineering and scientific disciplines, often as approximation to partial differential equations, which represent much of the continuum phenomena. Many forms of these equations are possible. Some of the applications are unsteady aerodynamics and aero elastic phenomena, visco-elasticity, visco-elastic panel in super sonic gas flow, fluid dynamics, electro dynamics of complex medium, many models of population growth, polymer rheology, neural network modeling, sandwich system identification, materials with fading memory, mathematical modeling of the diffusion of discrete particles in a turbulent fluid, heat conduction in materials with memory, theory of lossless transmission lines, theory of population dynamics, compartmental systems, nuclear reactors and mathematical modeling of a hereditary phenomena.

The theory of fractional differential equations has become an active area of investigation due to their applications in the fields such as physics, technical sciences and so on. One can see [1, 2, 5, 14, 15, 18, 20] and references therein.

The problem of existence of solutions of cauchy problem for fractional integro-differential equations was studied in numerous works; We refer the reader to the papers by Cuevas *et al.* [6, 7, 8], where they studied S-asymptotically w-periodic solutions. Recently, WANG and CHEN [22] considered a class of retarded integro-differential equations with nonlocal initial conditions where existence of solutions are given over a unbounded interval  $[0, \infty)$ .

Our main purpose in this paper is to extend a result from the above cited literature devoted fractional integro-differential equations with infinite delay. We provide sufficient conditions for the existence of the unique mild solution on a semiinfinite interval  $J = [0, \infty)$  for the fractional integro-differential equations (1) – (2) using the recent nonlinear alternative of Leray-Schauder type due to FRIGON and GRANAS for contraction maps in Fréchet spaces [11].

## 2. Preliminaries

We introduce notations, definitions and theorems which are used throughout this paper.

Let  $C([0, +\infty); E)$  be the space of continuous functions from  $[0, +\infty)$  into  $E$  and  $B(E)$  be the space of all bounded linear operators from  $E$  into  $E$ , with the usual supremum norm  $N \in B(E)$ ,  $\|N\|_{B(E)} = \sup\{|N(y)| : |y| = 1\}$ .

A measurable function  $y : [0, +\infty) \rightarrow E$  is Bochner integrable if and only if  $|y|$  is Lebesgue integrable. (For the Bochner integral properties, see the classical monograph of YOSIDA [23].)

Let  $L^1([0, +\infty), E)$  denotes the Banach space of measurable functions  $y : [0, +\infty) \rightarrow E$  which are Bochner integrable normed by

$$\|y\|_{L^1} = \int_0^{+\infty} |y(t)| dt.$$

In this paper, we will employ an axiomatic definition of the phase space  $\mathcal{B}$  introduced by HALE and KATO in [12] and follow the terminology used in [13]. Thus,  $(\mathcal{B}, \|\cdot\|_{\mathcal{B}})$  will be a seminormed linear space of functions mapping  $(-\infty, 0]$  into  $E$ , and satisfying the following axioms:

(A<sub>1</sub>) If  $y : (-\infty, b) \rightarrow E$ ,  $b > 0$ , is continuous on  $[0, b]$  and  $y_0 \in \mathcal{B}$ , then for every  $t \in [0, b]$  the following conditions hold:

- (i)  $y_t \in \mathcal{B}$ ;
- (ii) There exists a positive constant  $H$  such that  $|y(t)| \leq H\|y_t\|_{\mathcal{B}}$ ;
- (iii) There exist two functions  $K(\cdot), M(\cdot) : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  independent of  $y$  with  $K$  continuous and  $M$  locally bounded such that  $\|y_t\|_{\mathcal{B}} \leq K(t) \sup\{|y(s)| : 0 \leq s \leq t\} + M(t)\|y_0\|_{\mathcal{B}}$ .

(A<sub>2</sub>) For the function  $y$  in (A<sub>1</sub>),  $y_t$  is a  $\mathcal{B}$ -valued continuous function on  $[0, b]$ .

(A<sub>3</sub>) The space  $\mathcal{B}$  is complete.

Denote  $K_b = \sup\{K(t) : t \in [0, b]\}$  and  $M_b = \sup\{M(t) : t \in [0, b]\}$ .

**Remark 2.1.** 1. (ii) is equivalent to  $|\phi(0)| \leq H\|\phi\|_{\mathcal{B}}$ , for every  $\phi \in \mathcal{B}$ .

2. Since  $\|\cdot\|_{\mathcal{B}}$  is a seminorm, two elements  $\phi, \psi \in \mathcal{B}$  can verify  $\|\phi - \psi\|_{\mathcal{B}} = 0$  without necessarily  $\phi(\theta) = \psi(\theta)$ , for all  $\theta \leq 0$ .

3. From the equivalence of in the first remark, we can see that for all  $\phi, \psi \in \mathcal{B}$  such that  $\|\phi - \psi\|_{\mathcal{B}} = 0$ : We necessarily have that  $\phi(0) = \psi(0)$ .

We now indicate some examples of phase spaces. For other details we refer, for instance to the book by HINO *et al* [13].

**Example 2.2.** Let:

$BC$  the space of bounded continuous functions defined from  $(-\infty, 0]$  to  $E$ ;

$BUC$  the space of bounded uniformly continuous functions defined from  $(-\infty, 0]$  to  $E$ ;

$C^\infty := \{\phi \in BC : \lim_{\theta \rightarrow -\infty} \phi(\theta) \text{ exist in } E\}$ ;

$C^0 := \{\phi \in BC : \lim_{\theta \rightarrow -\infty} \phi(\theta) = 0\}$ , endowed with the uniform norm  $\|\phi\| = \sup\{|\phi(\theta)| : \theta \leq 0\}$ .

We have that the spaces  $BUC$ ,  $C^\infty$  and  $C^0$  satisfy conditions  $(A_1) - (A_3)$ . However,  $BC$  satisfies  $(A_1), (A_3)$  but  $(A_2)$  is not satisfied.

**Example 2.3.** The spaces  $C_g$ ,  $UC_g$ ,  $C_g^\infty$  and  $C_g^0$ .

Let  $g$  be a positive continuous function on  $(-\infty, 0]$ . We define:

$C_g := \left\{ \phi \in C((-\infty, 0], E) : \frac{\phi(\theta)}{g(\theta)} \text{ is bounded on } (-\infty, 0] \right\}$ ;

$C_g^0 := \left\{ \phi \in C_g : \lim_{\theta \rightarrow -\infty} \frac{\phi(\theta)}{g(\theta)} = 0 \right\}$ , endowed with the uniform norm

$$\|\phi\| = \sup \left\{ \frac{|\phi(\theta)|}{g(\theta)} : \theta \leq 0 \right\}.$$

Then we have that the spaces  $C_g$  and  $C_g^0$  satisfy conditions  $(A_3)$ . We consider the following condition on the function  $g$ .

$(g_1)$  For all  $a > 0$ ,  $\sup_{0 \leq t \leq a} \sup \left\{ \frac{g(t + \theta)}{g(\theta)} : -\infty < \theta \leq -t \right\} < \infty$ .

They satisfy conditions  $(A_1)$  and  $(A_2)$  if  $(g_1)$  holds.

**Example 2.4.** The space  $C_\gamma$ . For any real positive constant  $\gamma$ , we define the functional space  $C_\gamma$  by

$$C_\gamma := \left\{ \phi \in C((-\infty, 0], E) : \lim_{\theta \rightarrow -\infty} e^{\gamma\theta} \phi(\theta) \text{ exists in } E \right\}$$

endowed with the following norm  $\|\phi\| = \sup\{e^{\gamma\theta} |\phi(\theta)| : \theta \leq 0\}$ . Then in the space  $C_\gamma$  the axioms  $(A_1) - (A_3)$  are satisfied.

**Definition 2.5.** A function  $f : J \times \mathcal{B} \rightarrow E$  is said to be an  $L^1$  - Carathéodory function if it satisfies:

- (i) for each  $t \in J$  the function  $f(t, \cdot) : \mathcal{B} \rightarrow E$  is continuous;
- (ii) for each  $y \in \mathcal{B}$  the function  $f(\cdot, y) : J \rightarrow E$  is measurable;
- (iii) for every positive integer  $k$  there exists  $h_k \in L^1(J; \mathbb{R}^+)$  such that  $|f(t, y)| \leq h_k(t)$  for all  $\|y\|_{\mathcal{B}} \leq k$  and almost every  $t \in J$ .

The Laplace transformation of a function  $f \in L^1_{loc}(\mathbb{R}_+, E)$  is defined by

$$\mathcal{L}(f)(\lambda) := \hat{a}(\lambda) = \int_0^{\infty} e^{-\lambda t} f(t) dt, \quad \operatorname{Re}(\lambda) > \omega,$$

if the integral is absolutely convergent for  $\operatorname{Re}(\lambda) > \omega$ . In order to defined the mild solution of the problems (1) we recall the following definition

**Definition 2.6.** Let  $A$  be a closed and linear operator with domain  $D(A)$  defined on a Banach space  $E$ . We call  $A$  the generator of an integral resolvent if there exists  $\omega > 0$  and a strongly continuous function  $S : \mathbb{R}^+ \rightarrow B(E)$  such that

$$\left( \frac{1}{\hat{a}(\lambda)} I - A \right)^{-1} x = \int_0^{\infty} e^{-\lambda t} S(t) x dt, \quad \operatorname{Re} \lambda > \omega, x \in E.$$

In this case,  $S(t)$  is called the integral resolvent family generated by  $A$ .

The following result is a direct consequence of ([16], Proposition 3.1 and Lemma 2.2).

**Proposition 2.7.** *Let  $\{S(t)\}_{t \geq 0} \subset B(E)$  be an integral resolvent family with generator  $A$ . Then the following conditions are satisfied:*

- a)  $S(t)$  is strongly continuous for  $t \geq 0$  and  $S(0) = I$ ;
- b)  $S(t)D(A) \subset D(A)$  and  $AS(t)x = S(t)Ax$ , for all  $x \in D(A)$ ,  $t \geq 0$ ;
- c) for every  $x \in D(A)$  and  $t \geq 0$ ,

$$S(t)x = a(t)x + \int_0^t a(t-s)AS(s)x ds.$$

d) Let  $x \in D(A)$ . Then  $\int_0^t a(t-s)S(s)x ds \in D(A)$  and

$$S(t)x = a(t)x + A \int_0^t a(t-s)S(s)x ds.$$

In particular,  $S(0) = a(0)$ .

**Remark 2.8.** The uniqueness of resolvent is well-know (see PRÜSS [21]).

If an operator  $A$  with domain  $D(A)$  is the infinitesimal generator of an integral resolvent family  $S(t)$  and  $a(t)$  is a continuous, positive and nondecreasing function which satisfies  $\lim_{t \rightarrow 0^+} \frac{\|S(t)\|_{B(E)}}{a(t)} < \infty$  then for all  $x \in D(A)$  we have  $Ax = \lim_{t \rightarrow 0^+} \frac{S(t)x - a(t)x}{(a^*a)(t)}$ , see ([17], Theorem 2.1). For example, the case  $a(t) \equiv 1$  corresponds to the generator of a  $C_0$ -semigroup (see [4]) and  $a(t) = t$  actually corresponds to the generator of a sine family (see [3]). A characterization of generators of integral resolvent families, analogous to the Hille-Yosida Theorem for  $C_0$ -semigroups, can be directly deduced from ([17], Theorem 3.4). More information on the  $C_0$ -semigroups and sine families can be found in [4, 9, 10, 19].

**Definition 2.9.** A resolvent family of bounded linear operators,  $\{S(t)\}_{t>0}$ , is called uniformly continuous if  $\lim_{t \rightarrow s} \|S(t) - S(s)\|_{B(E)} = 0$ .

Let  $X$  be a Fréchet space with a family of semi-norms  $\{\|\cdot\|_n\}_{n \in \mathbb{N}}$ . We assume that the family of semi-norms  $\{\|\cdot\|_n\}$  verifies:  $\|x\|_1 \leq \|x\|_2 \leq \|x\|_3 \leq \dots$  for every  $x \in X$ . Let  $Y \subset X$ , we say that  $Y$  is bounded if for every  $n \in \mathbb{N}$ , there exists  $\overline{M}_n > 0$  such that  $\|y\|_n \leq \overline{M}_n$  for all  $y \in Y$ .

To  $X$  we associate a sequence of Banach spaces  $\{(X^n, \|\cdot\|_n)\}$  as follows: For every  $n \in \mathbb{N}$ , we consider the equivalence relation  $\sim_n$  defined by:  $x \sim_n y$  if and only if  $\|x - y\|_n = 0$  for  $x, y \in X$ . We denote  $X^n = (X|_{\sim_n}, \|\cdot\|_n)$  the quotient space, the completion of  $X^n$  with respect to  $\|\cdot\|_n$ . To every  $Y \subset X$ , we associate a sequence  $\{Y^n\}$  of subsets  $Y^n \subset X^n$  as follows: For every  $x \in X$ , we denote  $[x]_n$  the equivalence class of  $x$  of subset  $X^n$  and we defined  $Y^n = \{[x]_n : x \in Y\}$ . We denote  $\overline{Y^n}$ ,  $int_n(Y^n)$  and  $\partial_n Y^n$ , respectively, the closure, the interior and the boundary of  $Y^n$  with respect to  $\|\cdot\|_n$  in  $X^n$ .

The following definition is the appropriate concept of contraction in  $X$ .

**Definition 2.10** ([11]). A function  $f : X \rightarrow X$  is said to be a contraction if for each  $n \in \mathbb{N}$  there exists  $k_n \in (0, 1)$  such that:

$$\|f(x) - f(y)\|_n \leq k_n \|x - y\|_n, \quad \text{for all } x, y \in X.$$

The corresponding nonlinear alternative result is as follows:

**Theorem 2.11.** (*Nonlinear Alternative of Granas-Frigon, [11].*) Let  $X$  be a Fréchet space and  $Y \subset X$  a closed subset and let  $N : Y \rightarrow X$  be a contraction such that  $N(Y)$  is bounded. Then one of the following statements holds:

(C1)  $N$  has a unique fixed point;

(C2) There exists  $\lambda \in [0, 1)$ ,  $n \in \mathbb{N}$  and  $x \in \partial_n Y^n$  such that  $\|x - \lambda N(x)\|_n = 0$ .

### 3. Existence results

#### 3.1. Mild solutions

**Definition 3.1.** We say that the function  $y : \mathbb{R} \rightarrow E$  is a mild solution of (1) – (2) if  $y(t) = \phi(t)$  for all  $t \leq 0$  and  $y$  satisfies the following integral equation

$$(3) \quad y(t) = S(t)\phi(0) + \int_0^t S(t-s) f(s, y_s) ds, \quad \text{for each } t \geq 0.$$

We introduce the following hypotheses:

(H1) There exists a constant  $\widehat{M} \geq 1$  such that  $\|S(t)\|_{B(E)} \leq \widehat{M}$ , for every  $t \geq 0$ .

(H2) There exists a function  $p \in L^1_{loc}(J; \mathbb{R}_+)$  and a continuous nondecreasing function  $\psi : \mathbb{R}_+ \rightarrow (0, \infty)$  and such that:

$$|f(t, u)| \leq p(t) \psi(\|u\|_{\mathcal{B}}) \text{ for a.e. } t \in J \text{ and each } u \in \mathcal{B}.$$

(H3) For all  $R > 0$ , there exists  $l_R \in L^1_{loc}(J; \mathbb{R}_+)$  such that:  $|f(t, u) - f(t, v)| \leq l_R(t) \|u - v\|_{\mathcal{B}}$ , for all  $u, v \in \mathcal{B}$  with  $\|u\|_{\mathcal{B}} \leq R$  and  $\|v\|_{\mathcal{B}} \leq R$ .

Consider the following space

$$B_{+\infty} = \{y : \mathbb{R} \rightarrow E : y|_{[0,T]} \text{ continuous for } T > 0 \text{ and } y_0 \in \mathcal{B}\},$$

where  $y|_{[0,T]}$  is the restriction of  $y$  to the real compact interval  $[0, T]$ .

Let us fix  $\tau > 1$ . For every  $n \in \mathbb{N}$ , we define in  $B_{+\infty}$  the semi-norms by:  $\|y\|_n := \sup\{e^{-\tau L_n^*(t)}|y(t)| : t \in [0, n]\}$  where

$$L_n^*(t) = \int_0^t \bar{l}_n(s) ds, \quad \bar{l}_n(t) = K_n \widehat{M} l_n(t)$$

and  $l_n$  is the function from (H3).

Then  $B_{+\infty}$  is a Fréchet space with those family of semi-norms  $\|\cdot\|_{n \in \mathbb{N}}$ .

**Theorem 3.2.** *Assume that (H1)-(H3) hold, suppose that*

$$(4) \quad \int_0^{+\infty} \frac{ds}{\psi(s)} > K_n \widehat{M} \int_0^n p(s) ds, \quad n \in \mathbb{N}.$$

*Then the problem (1)-(2) has a unique mild solution on  $(-\infty, +\infty)$ .*

**Proof.** We transform the problem (1)-(2) into a fixed-point problem. Consider the operator  $N : B_{+\infty} \rightarrow B_{+\infty}$  defined by:

$$(5) \quad N(y)(t) = \begin{cases} \phi(t), & \text{if } t \leq 0; \\ S(t) \phi(0) + \int_0^t S(t-s) f(s, y_s) ds, & \text{if } t \in J. \end{cases}$$

Clearly, fixed points of the operator  $N$  are mild solutions of the problem (1)-(2). For  $\phi \in \mathcal{B}$ , we will define the function  $x(\cdot) : \mathbb{R} \rightarrow E$  by

$$x(t) = \begin{cases} \phi(t), & \text{if } t \leq 0; \\ S(t) \phi(0), & \text{if } t \in J. \end{cases}$$

Then  $x_0 = \phi$ . For each function  $z \in B_{+\infty}$  with  $z_0 = 0$ , we denote by  $\bar{z}$  the function defined by

$$\bar{z}(t) = \begin{cases} 0, & \text{if } t \leq 0; \\ z(t), & \text{if } t \in J. \end{cases}$$



If  $y(\cdot)$  satisfies (3), we can decompose it as  $y(t) = z(t) + x(t)$ ,  $t \geq 0$ , which implies  $y_t = z_t + x_t$ , for every  $t \in J$  and the function  $z(\cdot)$  satisfies

$$z(t) = \int_0^t S(t-s) f(s, z_s + x_s) ds \quad \text{for } t \in J.$$

Let  $B_{+\infty}^0 = \{z \in B_{+\infty} : z_0 = 0 \in \mathcal{B}\}$ . For any  $z \in B_{+\infty}^0$  we have

$$\|z\|_{+\infty} = \|z_0\|_{\mathcal{B}} + \sup\{|z(s)| : 0 \leq s < +\infty\} = \sup\{|z(s)| : 0 \leq s < +\infty\}.$$

Thus  $(B_{+\infty}^0, \|\cdot\|_{+\infty})$  is a Banach space. We define the operator  $F : B_{+\infty}^0 \rightarrow B_{+\infty}^0$  by:

$$(6) \quad F(z)(t) = \int_0^t S(t-s) f(s, z_s + x_s) ds \quad \text{for } t \in J.$$

Obviously the operator  $N$  has a fixed point is equivalent to  $F$  has one, so it turns to prove that  $F$  has a fixed point. Let  $z \in B_{+\infty}^0$  be such that  $z = \lambda F(z)$  for some  $\lambda \in [0, 1)$ . By the hypotheses (H1), (H2), we have for each  $t \in [0, n]$

$$\begin{aligned} |z(t)| &\leq \int_0^t \|S(t-s)\|_{B(E)} |f(s, z_s + x_s)| ds \\ &\leq \widehat{M} \int_0^t p(s) \psi(\|z_s + x_s\|_{\mathcal{B}}) ds \\ &\leq \widehat{M} \int_0^t p(s) \psi\left(K_n |z(s)| + (M_n + K_n \widehat{M} H) \|\phi\|_{\mathcal{B}}\right) ds. \end{aligned}$$

Set  $c_n := (M_n + K_n \widehat{M} H) \|\phi\|_{\mathcal{B}}$ . Then, we have

$$|z(t)| \leq \widehat{M} \int_0^t p(s) \psi(K_n |z(s)| + c_n) ds.$$

Then

$$K_n |z(t)| + c_n \leq c_n + K_n \widehat{M} \int_0^t p(s) \psi(K_n |z(s)| + c_n) ds.$$

We consider the function  $\mu$  defined by

$$\mu(t) := \sup \{ K_n |z(s)| + c_n \mid 0 \leq s \leq t \}, \quad 0 \leq t < +\infty.$$

Let  $t^* \in [0, t]$  be such that  $\mu(t) = K_n |z(t^*)| + c_n \|\phi\|_{\mathcal{B}}$ . By the previous inequality, we have

$$\mu(t) \leq c_n + K_n \widehat{M} \int_0^t p(s) \psi(\mu(s)) ds,$$

for  $t \in [0, n]$ . Let us take the right-hand side of the above inequality as  $v(t)$ . Then, we have  $\mu(t) \leq v(t)$ , for all  $t \in [0, n]$ . From the definition of  $v$ , we have  $v(0) = c_n$  and  $v'(t) = \widehat{M} K_n p(t) \psi(\mu(t))$  a.e.  $t \in [0, n]$ . Using the nondecreasing character of  $\psi$ , we get  $v'(t) \leq \widehat{M} K_n p(t) \psi(v(t))$  a.e.  $t \in [0, n]$ . Using the condition (4), this implies that for each  $t \in [0, n]$  we have

$$\int_{c_n}^{v(t)} \frac{ds}{\psi(s)} \leq \widehat{M} K_n \int_0^t p(s) ds \leq \widehat{M} K_n \int_0^n p(s) ds < \int_{c_n}^{+\infty} \frac{ds}{\psi(s)}.$$

Thus, for every  $t \in [0, n]$ , there exists a constant  $\Lambda_n$  such that  $v(t) \leq \Lambda_n$  and hence  $\mu(t) \leq \Lambda_n$ . Since  $\|z\|_n \leq \mu(t)$ , we have  $\|z\|_n \leq \Lambda_n$ .

Set  $Z = \{z \in B_{+\infty}^0 : \sup_{0 \leq t \leq n} |z(t)| \leq \Lambda_n + 1, \forall n \in \mathbb{N}\}$ . Clearly,  $Z$  is a closed subset of  $B_{+\infty}^0$ .

We shall show that  $F : Z \rightarrow B_{+\infty}^0$  is a contraction operator. Indeed, consider  $z, \bar{z} \in Z$ , thus using (H1) and (H3) for each  $t \in [0, n]$  and  $n \in \mathbb{N}$

$$\begin{aligned} |F(z)(t) - F(\bar{z})(t)| &\leq \int_0^t \|S(t-s)\|_{B(E)} |f(s, z_s + x_s) - f(s, \bar{z}_s + x_s)| ds \\ &\leq \int_0^t \widehat{M} l_n(s) \|z_s - \bar{z}_s\|_{\mathcal{B}} ds. \end{aligned}$$

Using (A<sub>1</sub>), we obtain

$$\begin{aligned} |F(z)(t) - F(\bar{z})(t)| &\leq \int_0^t \widehat{M} l_n(s) (K(s) |z(s) - \bar{z}(s)| + M(s) \|z_0 - \bar{z}_0\|_{\mathcal{B}}) ds \\ &\leq \int_0^t \widehat{M} l_n(s) K_n |z(s) - \bar{z}(s)| ds \\ &\leq \int_0^t \left[ \bar{l}_n(s) e^{\tau L_n^*(s)} \right] \left[ e^{-\tau L_n^*(s)} |z(s) - \bar{z}(s)| \right] ds \\ &\leq \int_0^t \left[ \frac{e^{\tau L_n^*(s)}}{\tau} \right]' ds \|z - \bar{z}\|_n \\ &\leq \frac{1}{\tau} e^{\tau L_n^*(t)} \|z - \bar{z}\|_n. \end{aligned}$$

Therefore,  $\|F(z) - F(\bar{z})\|_n \leq \frac{1}{\tau} \|z - \bar{z}\|_n$ . So, the operator  $F$  is a contraction for all  $n \in \mathbb{N}$ . From the choice of  $Z$  there is no  $z \in \partial Z^n$  such that  $z = \lambda F(z)$ ,  $\lambda \in (0, 1)$ . Then the statement (C2) in Theorem 2.11 does not hold. The nonlinear alternative of Frigon and Granas shows that (C1) holds. Thus, we deduce that the operator  $F$  has a unique fixed-point  $z^*$ . Then  $y^*(t) = z^*(t) + x(t)$ ,  $t \in (-\infty, +\infty)$  is a fixed point of the operator  $N$ , which is the unique mild solution of the problem (1) – (2).

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