The Singapore mathematics curriculum and implications for adopting and developing curricula in the U.S.

By Scott Baldridge

Singapore's elementary mathematics curriculum is by no means ordinary. In a coarse way the data supports this: the Trends in International Mathematics and Science Study[1] (TIMSS) rated Singapore's elementary students the best in the world in mathematics [Gonzales, et al, 2005]. William Schmidt, the director of the U.S. National Research Center for TIMSS, suggests that the curriculum has something to do with this success. He writes that in top achieving Asian countries the mathematics programs are focused and rigorous, and that the curricula are coherent [Schmidt, et al, 2002]. Unlike Singapore, there is no single math curriculum in the U.S., and he characterizes those that do exist as "a mile wide and an inch deep." Furthermore, he found that such curricula are unfocused, repetitive, and unchallenging [Cogan, Schmidt, 1999].

The words "focused, rigorous, coherent" bring to mind curricula that lead students through a sequence of increasingly complex topics and performances over the grades, which reflects the logical and sequential nature of knowledge in mathematics. Yet these technical descriptions are inadequate for describing just how impressive the top countries are at accomplishing this feat in their curricula. In this paper I propose that it is appropriate to think of coherent Asian curricula metaphorically as *stories*: reading through an Asian curriculum feels like reading a well-written, nonfiction, illustrated story.

The term `story' suggests a particular structure to the organizational quality of curriculum writing as well as a certain aesthetic quality that is clearly missing in many curricula written in the U.S. The storybook quality of the Singapore curriculum is also strongly present in the Japanese, Korean, and Hong Kong curricula. These curricula have their strengths and in some cases are better than the Singapore curriculum at presenting a coherent story. In this article I will use the Singapore textbooks as the model curriculum simply because it is readily available in the U.S.[2], already written in English, and the one with which I am most familiar. Throughout this article Singapore curriculum refers specifically to the *Primary Mathematics U.S. Edition Textbooks* for grades one through six, and the seventh and eighth grade textbooks called *New Elementary Mathematics Syllabus D.*

In the first part of this paper I will outline the storybook nature of the Singapore curriculum and how it is distinctly different from the two mainstream types of U.S. curricula. Because the gulf is so wide between Asian and U.S. curricula, in the second half I will discuss some possible ways the U.S. might fund and develop textbooks that can bring some of the desirable qualities of the Asian curricula to the U.S.

Artfully Written Stories

Elementary curricula from Asia have many of the same characteristics as artfully written stories. They develop `characters' and weave `storylines' together until a crystal clear image of arithmetic and geometry emerges after eight years. The stories are not about fictional characters or issues—they do not begin with, ``Susan was making some cookies…'' Rather, the `story' is about mathematical issues. In arithmetic it is the saga of the `maturing' concept of number, i.e., how numbers begin as whole numbers in first grade and naturally develop into real numbers by seventh grade. Geometry is a nonlinear saga about the emergence of an axiomatic system given through a series of `flashbacks' that show the need for fewer and fewer axioms and undefined terms (for example, a triangle is an undefined term in the beginning).

Many elementary curricula written in the U.S. also follow a similar "whole numbers to real numbers" development. However, it is in the way these curricula develop that make them so different from

[1] TIMSS is a multinational study that includes elementary and secondary student achievement. The textbooks described in this article were used by Singapore students when they placed first in TIMSS.
[2] See singaporemath.com.

Singapore's. At best, these curricula read like a series of semi-related self-contained episodes (booklets); at worst, they read like a loosely grouped set of unrelated topics (encyclopedic).

The Singapore curriculum is written so that each new chapter draws deeply from the content learned in the previous chapters and years, weaving the knowledge learned into a coherent whole. This produces an effect similar to reading a good novel: The storyline, even after weeks of not reading, is easy to pick up again because the novel pulls the reader back into the plot immediately—there is no need to review previous chapters because the plot brings out and adds to what has already happened. This cumulative aspect of the Singapore curriculum's plot, along with its themes, character development, and composition, reflect some of the characteristics of a well-written story.

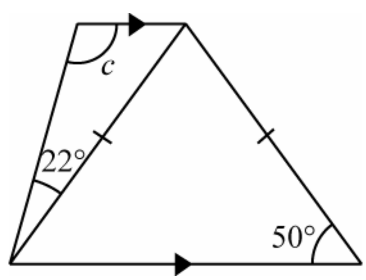
Themes. An important characteristic of the Singapore curriculum is its themes—ideas that pervade throughout elementary school. For example, the themes of arithmetic are place value and units. Much has been said in recent articles and textbooks by mathematicians (Aharoni, Howe, Parker/Baldridge) about the need to make place value a prominent theme in U.S. elementary curricula. There is a good reason for this: it is important for developing flexibility with numbers (mental math and estimation), it is a prerequisite for all of the standard algorithms, it is necessary for explaining the relationship between decimals and rational numbers, and it plays a crucial role in understanding decimal expansions and how to locate them on a number line. The textbooks from Singapore emphasize and build on place value in precisely these ways. The books are filled with numerous problems that test and develop understanding of place value; the problems are incorporated throughout the curriculum and are strategically positioned to help develop proficiency and understanding of arithmetic.

The Singapore curriculum focuses intensely on how to make and use units. All curricula talk about units to some degree, the difference here is that the formation and use of units is keyed to almost every explanation. Usually units bring to mind measurement, like an inch or a meter, but the Singapore curriculum includes more abstract forms. For example, whole numbers decompose into different units—ones, tens, hundreds, thousands—important for place value. The textbooks show how focusing on different place value units helps students solve problems like 7163×2, which can be easily answered by doubling 7 thousands 16 tens 3 ones. (This shows that place value is actually part of the `making and using units' theme.)

One important reason for focusing on units in the Singapore curriculum is to solidify fractions as numbers. Initially, fractions are introduced as adjectives (half a cup of milk) and operators: ¾ means break into four equal pieces and take three. But very soon thereafter in the curriculum students learn to count, add, and subtract with fractional units. When the focus is on units, fractions arithmetic arises as a natural extension of whole number arithmetic: 2 *fifths* + 2 *fifths* = 4 *fifths* is just an extension of 2 apples and 2 more make 4 apples. That, in turn, helps students to think of fractions abstractly as numbers.

In geometry, the Singapore theme is logical deduction. Starting in fourth grade, students work through picture proofs and use measurements to understand concepts like, ``the sum of interior angle measures of a triangle is 180 degrees.'' Initially, every such fact is taken as an axiom. Once there are enough such facts, students apply them to ``find the unknown angle'' problems. For example, by fifth grade the Singapore curriculum builds to 11 facts about parallel lines and triangles and then asks questions like:

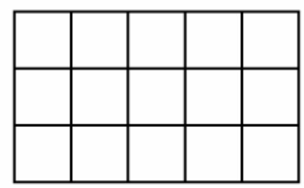
Find the unknown marked angle [arrows denote parallel lines, tick-marks denote congruence].



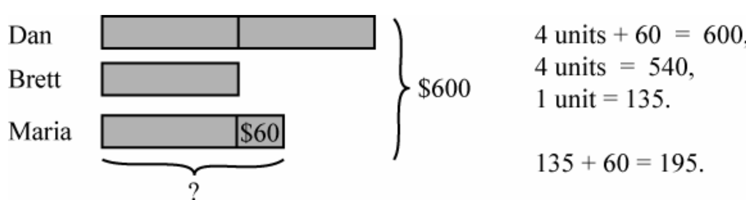
This is a simple enough problem, but solving it requires students know the isosceles triangle theorem, alternate interior angles theorem, and that there are 180 degrees in a triangle (or on a line). As a deduction problem, the only difference from it and a U.S. tenth grade problem is how the tenth grade problem is stated: ``Prove that angle *c* is 108 degrees.'' In seventh grade, the Singapore curriculum starts over with a reduced set of axioms (and fewer undefined terms) and proves the remaining facts from the reduced axiom list. Students do more ``find the unknown angle'' problems (which by then often require auxiliary lines) and guided proofs that step them through the proofs of important theorems. By eighth grade Singapore students are proving theorems on their own, a full four years after the first deductive geometry problems were introduced and two years before U.S. students generally do any serious deductive geometry.

Characters. In arithmetic the main characters are whole numbers, rational numbers, integers, and real numbers; and in geometry the central characters are segments, lines, triangles, rectangles, circles, and cubes. The narrative brings out the personality traits of the main characters. For example, a trait of decimal numbers is that they can be easily parsed by place value into useful denominations, as when we write 1234 as 123 tens 4 ones. A personality trait of (proper) rational numbers is that they seemingly add in a different way than whole numbers do.

Often one character plays a supportive role in helping to explain and justify a personality trait of another character. A simple but eloquent example of this interaction is the use of a rectangular array of squares to justify a very important personality trait of whole numbers: 3+3+3+3+3 = 5+5+5. In fact, the geometric characters (segments, rectangles, circles, etc.) are used to create almost all picture models used in the Singapore curriculum. In this context a model is just a visual way to represent numbers, just as a graph is a visual way to represent a function in calculus. In the Singapore curriculum, important models include numbers on a line, chip models, and rectangular arrays. A particularly powerful model used almost exclusively by the Singapore curriculum is the bar diagram. Fifth grade students use it to solve algebra problems long before they do so with variables in seventh grade:


A rectangular array

Dan saved twice as much as Brett. Maria saved $60 more than Brett. If they saved $600 altogether, how much did Maria save?



Maria saved 195 dollars.

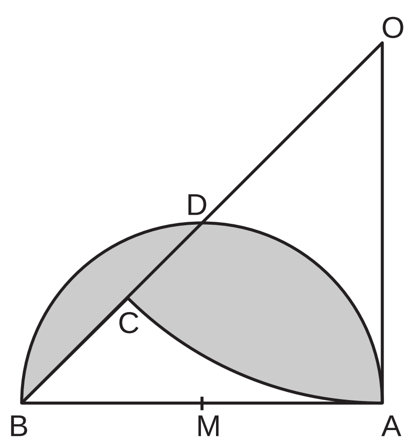
The Singapore curriculum is filled with explanations that use bar diagrams like the one above; the bars are used to represent fractions, decimals, ratios, rates, percents, etc. Bar diagrams are used in Singapore to introduce prealgebra and algebra problems in fourth and fifth grade in a way that doesn't require the use of variables. By the time Singapore students reach seventh grade, they are already familiar with these types of problems and are ready for the `letters only' solutions that work for all numerical cases.

Singapore's use of geometry characters to model arithmetic and word problems differs in a fundamental way from most curricula written in the U.S. The Singapore curriculum uses only a small set (<10) of different picture models for the entire elementary curriculum. Furthermore, the hands-on models (called manipulatives) tend to be common objects found around the house like tape measures, liter jars, paper clips, etc. Over the years students develop increasing familiarity with this limited set of consistently-used models, which may make it easier for students to concentrate on the mathematics instead of spending time on learning each new model.[3] Dramatically opposite this practice, prospective and working teachers in the U.S. are constantly bombarded with messages claiming that math lessons should be centered on manipulatives or models designed specifically for each lesson. A whole industry exists in the U.S. with thousands of different manipulatives/models for teachers to buy and use in their classrooms. Many curricula encourage the use of `a different model for each lesson' because, not incidentally, their publishers are also in the business of selling the same plastic knickknacks.

The main point, of course, is that the Singapore curriculum is completely and utterly devoted to presenting mathematics in much the same way a good story is devoted to its main characters. Models in Singapore are certainly an important part of that presentation, but they play only supportive roles.

Plot. Generally a story develops its plot through a series of increasingly serious problems or events resulting in a final conflict-resolution. The quintessential example of a plot line in mathematics is Euclid's Book I and his build up to the proof of the Pythagorean Theorem. The Singapore curriculum also has a plot: Below is one of the final "find the unknown area" problems from the seventh grade textbook.

In the figure, AC is an arc of a circle, center O. BDA is a semicircle, center M. If OA is perpendicular to BA and OA=BA=14cm, find the total area of the shaded parts. (Take pi to be 22/7)



This problem is minimally a six step problem.[4] Here's the count: find angle O, use it to find sector area of AC, find the area of triangle BAO, subtract to find the area of the unshaded region under the semicircle, find the area of the semicircle, and subtract the unshaded region from it to find the area of shaded region. Notice that the problem requires knowledge of right isosceles triangles, areas of circles, sectors, triangles, and how to compute areas of composite regions. These topics are covered separately in grades 4-6 with four- and five-step problems in sixth grade, three- and four-step problems in fifth grade (like the unknown angle problem above), and two- and three-step problems in fourth grade. This seventh grade problem comes near the end of a series of increasingly difficult problems—the plot thickening with each grade.

An engaging story, even if it is nonlinear with flashbacks, builds the plot towards the final conflict-resolution. Every scene, every event, every twist, every problem, moves the plot forward. It is the plot-building aspect of a good story that helps the reader keep the entire story in mind as he or she reads. Authors accomplish this through careful attention to every scene, constantly asking, "How does this scene or event move the plot forward?"

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[3] This observation was made by Thomas Parker and presented at MSRI's, "The Mathematical Knowledge for Teaching (K-8): Why, What and How?" conference in May, 2005.
[4] Multistep problem in this article means that the student must choose the steps to make in solving the problem. Guided problems, common in many newer curricula, ask a series of questions guiding the student through the problem or set of tasks.

One way the Singapore curriculum accomplishes this task is through *juxtaposition*. The Singapore curriculum never repeats a topic; each topic is expected to be taught to mastery before moving on to the next topic.[5] After a topic is covered, however, it becomes part of the overall plot, and is woven back into the story several times in new ways that also move the plot forward. This is most often done by juxtaposing a new topic with one (and usually only one) previous topic. The old topic, when it was first mastered, was the *abstract* concept which students were stretching to grasp. Later, when juxtaposed with a new topic, it becomes the *concrete* example or basis for understanding the new topic. For example, in fourth grade, Singapore curriculum banks on the students' *mastery* of the whole number multiplication algorithm learned in third grade to *explain* how to multiply decimal numbers by a whole number (using chip models as an additional link between the two topics). This gives students enough skill to work many real-world word problems involving decimal multiplication, where they begin to see the usefulness of decimal numbers as well as develop a deeper appreciation of multiplication. The problems, solved by extending the whole number multiplication algorithm, serve as a feedback-loop, solidifying students' understanding of algorithms and arithmetic, and thus making straightforward review of the whole number multiplication algorithm much less necessary.

As in the example above, how the topics are woven back into the curriculum deepens students' initial understanding of them. One can liken this to a plot twist in a novel where the reader finds out that an evil character was really good all along: such a plot twist forces the reader to go back through the story in his or her mind and *change* how they perceived *all* earlier scenes and events. Similarly, the recursivity of Singapore's curricular structure leads to what I will call a continual *retrospective re-cogitative plotline.* (Compare this to the concept of a spiral curriculum [cf. J. Bruner, 1960].)

Imagine creating a curriculum where these links are occurring in each section of every textbook, with every new idea juxtaposed with an old one; old and new feeding off and enriching each other. Imagine, further, that these links thread through the entire curriculum so that no idea can be learned in isolation, or in a way that doesn't affect understanding of what was previously learned. If it also leads students through a sequence of increasingly complex problems over the grades, such a curriculum is coherent in the way that Asian curricula are coherent. If, in addition, it is written well with a retrospective re-cogitative plotline, then it is a coherent story in the same vein as Singapore's.

Composition. A well written story avoids clutter and distraction, has smooth transitions between chapters, and is written so that every word entices the reader onward and contributes to the understanding of the plot. In a mathematics curriculum these characteristics translate into books that contain mathematics and nothing but mathematics, with clean presentations clearly linking one topic to another, and explanations that are simple and concise.

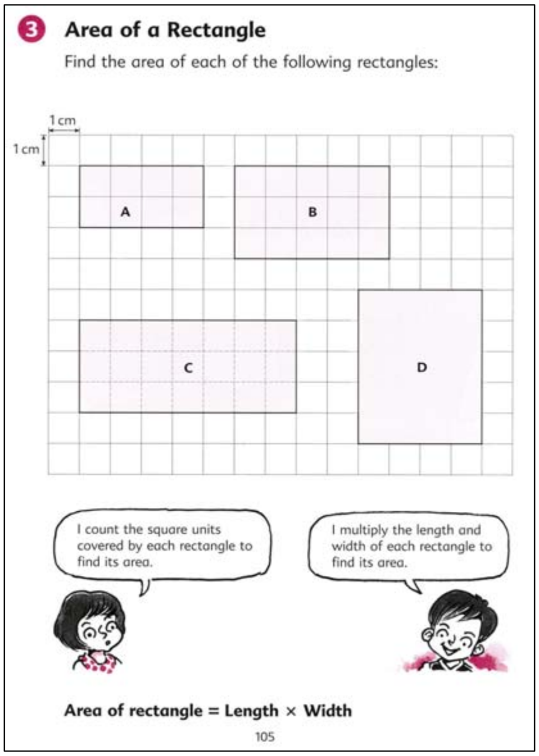
The layout of the book is very important to any good story. For example, imagine a layout of Charlotte's Web where every page is filled with extraneous pictures of exotic zoo animals, and each chapter contains several side stories of Wilbur and other farm animals in bright green boxes. Imagine biographies of E.B. White and others who were involved in the animated film version scattered throughout each chapter. Imagine "*Did you know?*" boxes on each page that provide comments and insights by E.B. White about his story. With only enough room for a paragraph or two of the story per page, imagine how hard it would be to read Charlotte's Web. Singapore's layout focuses students' attention on the mathematics in the same way layouts of good illustrated children's stories focus students' attention on the story.

The Singapore textbooks do not contain distracting extras such as long introductions and summaries. Biographical stories, explorations, or discussions of non-mathematical topics are not included until seventh grade, and then only prudently. Additional homework is relegated to workbooks while group projects, explorations, and applications to other subjects are put in separate teacher guides, and there are special workbooks for remedial and for advanced students. The pictures effectively convey meaning; they are not stylistic extras. The judicious use of white space makes the textbooks easy and enjoyable to read, and

[5] Many curricula in the U.S. spend the first three months of every school year reviewing last year's material.

keeps students focused on learning mathematics. To see the difference pick up almost any other elementary textbook: they will feature distracting side-bar messages, unnecessary drawings, showy photographs and highlighted boxes, and frequent font changes.[6] The mathematics in these textbooks is buried under the weight of the layout.

Skimming briefly through the Singapore textbooks one finds that chapters are linked roughly through their prerequisites. That makes sense—a chapter on multiplying by a fraction should come before a chapter on the formula for the area of a triangle. Chapters in the Singapore textbooks are also linked through other, more subtle, devices. For example, the first three chapters in Primary 6B are (1) fraction division, (2) area of a circle, and (3) pie charts—three seemingly different topics. It is interesting that the Singapore curriculum uses rectangles to represent fractions, except in the fraction division chapter where sectors of circles do double duty as preparation for circle area (the next chapter). The third chapter has students thinking about sectors of circles to construct pie charts, a topic usually clumped together with bar charts and scatter plots in other textbooks under the heading ``representing data graphically.'' The three chapters flow together naturally because students are using the circle model daily for the duration of the topics.



Finally, explanations in the Singapore textbooks are simple and concise. They are kept short by using pictures of cartoon children that announce key ideas in one sentence dialog balloons (using a layout that is reminiscent of Dr. Seuss stories). For example, the third grade textbook introduces the formula for the area of a rectangle by displaying four rectangles, two cartoon children, and four child-friendly sentences. The area units are visible in the first of the four rectangles, but `fade away' by the last rectangle. The first kid says, ``I count the square units covered by each rectangle to find its area.'' while the second kid answers smartly, ``I multiply the length and width of each rectangle to find its area.'' Underneath the children is the statement, ``Area of rectangle = length × width,'' linking the definition of area given in the previous pages with the formula presented. That's it—simple, clear, concise.

Careful attention to tight composition keeps the Singapore textbooks short. Each grade contains 10 to 15 topics covered in about 200 pages (1 page a day). The short length helps teachers and students focus on what is important. In this setup, every page is meant to be taught to mastery, and skipping a chapter would be like skipping a chapter in a novel.

Textbooks in the U.S.

Current textbooks in the U.S. generally fall into one of two curriculum-type categories: encyclopedic or booklet-based. Neither category has many storybook features. The encyclopedic textbooks bear no resemblance to stories, and while each booklet in a booklet-based curriculum may be coherent, the curriculum fails to tell a coherent story over the years. Below are some specific comments about each curriculum type.

[6] It's not clear why publishers taint textbooks in this way. Rarely have I met a practicing elementary teacher who finds this desirable.

Encyclopedic. Textbooks in this category tend to cover many topics only superficially. Typically each grade-level textbook starts with problem solving or the addition algorithm and ends 650 pages and 30 chapters later with some geometry [Schmidt, et. al. 1999]. Cuoco describes these curricula as *flat*: all 30 topics are presented as if roughly equal in importance [Cuoco, 2001]. This flatness makes it difficult for teachers to focus on what is important for their grade so that when (as is usually the case) they are pressed for time and start skipping topics, they are as likely as not to skip the very topics important for their students to learn. Indeed, TIMSS suggests that most often ignored topics in elementary mathematics in the U.S. include measurement and geometry.[7]

Models are often poorly used in encyclopedic textbooks. For example, there is a very nice one-to-one correspondence between the visual steps presented in the chip model and the pencil marks one makes while performing the standard addition algorithm. It is still easy to find elementary curricula that do not use the chip model (or any model) to explain the standard algorithms—not using the model is equivalent to not illustrating the derivative in calculus using a tangent line to the graph of a function at a point. When a textbook does use the model, it is often for a single explanatory case. Compare that to the first semester third grade Singapore textbook that has room in 10 pages to display 14 chip models for every relevant case needed to explain the addition and subtraction algorithms.

One of the egregious aspects of some encyclopedic textbooks is the pandering they do to school districts' fears of standardized tests. Publishers have concluded that a textbook is more marketable if it includes pages on how to "beat the test," displaying techniques like "eliminate wrong answers." Even if test taking skills should be part of elementary mathematics textbooks, the actual problems found on these pages are often so straightforward and simple that a child has absolutely no reason to "make smart choices," like in the *fourth grade* problem below:

> Which sum is the same as the difference between 14 and 6? a. 7+1, b. 10+10, c. 5+4, d. 14+6.
>
> —Test Prep Strategy, Math Grade 4, Scott Foresman-Addison Wesley.

Such pages are certainly not about learning mathematics. In fact, "test prep" pages are more likely to teach children that learning mathematics is not nearly as important as learning how to pass standardize tests.

Booklet-Based. Curricula of this type come in the form of several booklets used either by the teacher or the students.[8] The booklets are generally self-contained, take about a month, and cover a given topic such as using area to plan a city park. The very best booklets tell short stories about a given topic, but these short stories still do not succeed in building a cumulative storyline during the school year and throughout the grades.

Booklet-based curricula can suffer some of the same problems as encyclopedic textbooks. For example, one booklet-based curriculum has 8 booklets in a grade, each with 72 to 96 pages for a total of 673 pages (if indexes, table of contents, glossaries are excluded, almost 600 content pages). Each booklet has 5 to 7 sections for a total of 50 sections. While an individual booklet looks like an Asian textbook for a single grade, taken altogether the set of 8 booklets is equivalent in size to one encyclopedic textbook.

Each booklet stays within a given strand (number, measurement, geometry, etc.). The minimal set of prerequisites per booklet in and between grades means they can be taught in many different orders. This flexibility is touted as one of their selling points, but it is this very feature that prevents the development of problems like the Singapore area problem above. For example, the writer of a booklet on the area of triangles cannot assume that students have worked through a booklet on angles in a triangle; therefore he

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[7] U.S. students are above average at rounding on the TIMSS exam, but below average in measurement and geometry [Schmidt, et. al. 1999].
[8] Some curricula do not have textbooks for students.

cannot reasonably ask two-step problems that involve both ideas.[9] Interesting and multifaceted problems are eliminated from the curriculum simply due to the way the curriculum is *structured.*

Implications for Adopting Mathematics Curricula in the U.S.

Judging from the popularity of the Singapore curriculum amongst parents and private schools, there is a great need for story-based curricula in the U.S. Already there are several mathematicians and educators working hard (with little funding) to help school districts implement U.S. editions of the Singapore curriculum in the U.S. High quality translations of Japanese textbooks are also now available. One of the main impediments to implementing U.S. versions of Asian curricula is the state adoption processes and the politics needed to get curricula adopted. The U.S. would greatly benefit from encouraging each state to approve story-based curricula for adoption and use in public schools. There are many different story-based curricula available from different countries—some translated, some not. Below is a list of story-like curricula written in English that I have found (based upon years of studying curricula) to be particularly well designed:[10]

- Primary Mathematics U.S. Edition Textbooks and Workbooks, Grades 1—6, from Singapore,
- Mathematics for Elementary School, Grades 1—6, from Japan,
- New Elementary Mathematics Syllabus D, Grades 7—10, from Singapore, and
- Pleasurable Learning Mathematics, Grades 7—9, from Hong Kong.

The first series is the (famous) Singapore elementary curriculum discussed in the first part of this article. It is ready for state adoption except for the small caveat that has hurt it chances in the past: the only topics in K-6 state standards not covered in the Primary Mathematics series are simple probability, stem-and-leaf plots, median, and mode—a couple weeks worth of material which can be easily supplemented. In the past this was used by bean-counting administrators as one reason to oppose implementing it in their schools (cf. [Garelick 2006]). Fortunately, a new version (the U.S. Standards Edition) will be available soon which includes these topics without interfering with the integrity of the existing storyline, making it a near-perfect curriculum of its type for implementing in the U.S.

The second series, Mathematics for Elementary School, is a delightfully compact set of mathematics textbooks from Japan that thoughtfully develops measurement and geometry. (This curriculum cleverly incorporates origami to justify geometry facts, for example.) The mathematical explanations are very well written. The Japanese textbooks do not use bar diagram problems or unknown angle problems as effectively as the Singapore textbooks, and maybe because of that, there are fewer confidence-boosting word and geometry problems—the type of problems needed to build to the interesting seventh grade problems like the Singapore problem described above. Still, the story-book nature of this curriculum is undeniable and is another nice example of coherent curriculum.

The New Elementary Mathematics series is a middle school curriculum in Singapore meant to follow the Primary Mathematics textbooks. The real strength of this curriculum is in its problems. There are literally thousands of interesting problems to solve per grade, from algebra and unknown angle problems to proofs, from skill-building to real-world application problems using coordinate geometry. There is even a healthy set of quirky brain-teaser problems that have been so popular in the U.S. over the past fifteen years. The problems are carefully sequenced to produce a cumulative effect similar to reading a good story. These textbooks do lack the punchy prose of Singapore's elementary textbooks, but the mathematics problems more than make up for this fault, showing that middle school does not have to be a wasteland of mediocrity and review.

When it comes to naming a story-based curriculum for teaching deductive geometry, it just does not get better than Hong Kong's Pleasurable Mathematics series for middle school. The explanations and activities

[9]That is, unless the author's are willing to fix a strict order on the booklets, which runs contrary to using the booklet format.
[10] Please see the references for complete information on where to find these textbooks.

are superbly written; the problems and proofs are challenging yet at a level meant for all students (in a way that boosts confidence instead of destroying it); and the introduction to axiomatic geometry is gentle and innocuous. For example, instead of spending weeks proving pedantic statements, the books announce and justify a few important facts (axioms) and use those facts to get students started on interesting deductive geometry problems as quickly as possible. The textbooks are also careful not to bog students down with heavily structured proof formats. Like the New Elementary Mathematics curriculum, the Pleasurable Learning textbooks are pretty much done with Euclidean geometry (as it is usually taught in U.S. K-12 schools) by eighth grade.

The Pleasurable Learning Mathematics curriculum is so good that Thomas Parker and I initially chose it as the model curriculum for our second textbook for prospective teachers, *Elementary Geometry for Teachers* [Parker-Baldridge, 2006]. Our textbooks for prospective teachers are unique among mathematics textbooks in that we require students study a model school mathematics curriculum[11] on a daily basis in an effort to get them ready to teach from *any* curriculum (cf. [McCrory, 2006]). Our first textbook used the Primary Mathematics textbooks from Singapore for mathematics found in Grades 1—6, but we needed a middle school curriculum for the second textbook on geometry. In many ways the Pleasurable Learning curriculum is the perfect compliment to the Primary Mathematics curriculum in its problems, tone, and punchy style, but unfortunately the textbooks are hard to acquire in the U.S. (although they could be if the U.S. market demanded it). For this reason we chose the New Elementary Mathematics series for its excellent problems and used the Pleasurable Learning curriculum for inspiration when writing the second textbook.

It should be noted that none of the four curricula are ``Teacher-proof'' (cf. [Garelick, 2006]). Each requires knowledgeable teachers who understand the mathematics being presented and have appropriate pedagogical expertise as storytellers to ``bring the stories to life'' (cf. ``actors in a play'' from [Ma, 1999]). Training and a healthy teaching culture is also necessary to help teachers change their teaching practices to reflect teaching in a curriculum where mastery is expected.

Three Current Models for Developing Mathematics Curricula in the U.S.

With the release of the new Curriculum Focal Points publication by the NCTM there will surely be a call for new curricula in the U.S. that follow its recommendations.[12] In fact, the stated purpose of the document is to provide "a framework on which the next generation of state and district-level mathematics curricula might be built" [NCTM, 2006]. Story-based curricula fit nicely into this new framework; this type of curriculum appears to be an idea whose time has come. But it is not clear that current models employed in the U.S. to write curricula will suddenly produce story-based curricula. I have personal experience with three models used in the U.S.: curricula created by publishers, curricula funded by grants, and individuals writing textbooks on their own. This section is about those experiences.

About a year ago I was contacted to write a section of a textbook for a well-known publisher. Here is how that textbook was to be written: three or four people were selected to be the overseers of the textbook. They would be called the "authors" but not do much of the initial writing. Instead, they fleshed out the topics to be written; so for example, they told the publisher that they needed a section on multiplication of fractions. Next the publishers contacted authors like me to write the needed sections. In my case, I was asked to write a fifteen page section on equalities and inequalities in about a month. For my work I would receive $5,000 as a ghost writer. In the end I did not agree to the terms and did not write the piece. If I had, the overseers would have edited the material, included it with tens of other authors' sections into one binding, and called it a textbook.

This model is relatively cheap, efficient, and probably very profitable for the publishers, but very problematic for producing story-based curricula. One can certainly see why so many textbooks end up looking like encyclopedias! Even if each individual author did an outstanding job writing and editing their

[11] Neither my coauthor nor I make any money from the sales or distribution of any elementary textbook, nor are we invested in any company that does.
[12] See http://nctm.org/focalpoints.

section, they would have no idea what material was presented before it or after it in the textbook. How could the overseers ever put together a coherent picture of mathematics throughout the textbook and over several grades without significantly re-writing each section? I suspect that the publishing company also put considerable pressure on the overseers not to throw-away or change much of the author-written sections, because re-writing or asking for a new ghost writer gets prohibitively expensive fast while significantly increasing time to production.

My experience with the second model of curriculum development comes from working on the Connected Mathematics Project (CMP), an NSF funded mathematics curriculum funded in the 1990's. The project received $4.9 million dollars to develop and disseminate a middle school mathematics curriculum, and recently acquired another $4.6 million to re-write the first edition. Like many of the NSF funded curricula of the 1990's, it was booklet-based and adhered to the NCTM standards.

CMP ran like a typical grant proposal project. After winning the grant award, the five investigators hired a staff of coordinators, teachers, and graduate students to help in the writing and editing of the booklets. Each booklet was assigned one or two investigators who oversaw the development and writing of the booklet. The investigators wrote much of the booklet and farmed out the development of homework problems, illustrations, and teacher guides to the graduate students and staff. As materials were created they were given to teachers to try with students in the classroom, and the feedback was used to edit and revise the booklet until it was ready for production.

This model is definitely an improvement over the publisher-driven one. But writing in this environment still presents potential hurdles to creating a story-based curriculum. For example, the act of dividing up booklets (or chapters) can cause investigators to focus too narrowly on their assignments. In some ways it is like writing for a television sitcom: each episode takes on some problem that is dealt with within a 30 minute period, with little reference to what has happened in other episodes.

The last model for textbook/curriculum writing is the lone author (or pair of authors) who write entirely by themselves. As a coauthor of textbooks for prospective teachers [Parker-Baldridge, 2004, 2006], it is also the model with which I have the most familiarity. The writing process for this model most closely resembles that of a novelist: long toiling hours of plot formation, adherence to themes, attention to character development, and seemingly endless writing and re-writing—a real labor of love. This model is commonly used in graduate-level mathematics; virtually all upper undergraduate textbooks and all graduate textbooks are written in this way. Maybe it is not so surprising that many of the most popular mathematics textbooks at these levels also have story-like characteristics.

There are three stumbling blocks with this model. The first downside to using it is time. Books written in this way can easily take five to ten years because a mathematician-author often re-writes large portions of the textbooks each time he or she teaches the course. Secondly, the model requires broad institutional support. For example, the infrastructure exists in research-level universities to support mathematicians in the creation of undergraduate and graduate textbooks, but very little support exists for mathematician-authors to get involved with the development of elementary curricula. The third issue is that, like novels, the quality of a textbook greatly depends on the author's ability to write for their intended audience.

Implications for Developing Mathematics Curricula in the U.S.

In this article we have explored Singapore's story-based curriculum and why it differs so sharply from encyclopedic and booklet-based curricula found in the U.S. We have also described three models for how curricula are currently developed in the U.S. and looked at possible reasons these models have not resulted in elementary curricula with story-like characteristics. In this last section we discuss two possible models for developing story-based curricula in the United States.

Grant Driven Model. Analysis of the Singapore curriculum leads me to offer a different model than the three above for developing curricula. This model is based, in-part, on discussions I have had with curriculum developers from Japan on how they developed the curriculum mentioned above. A key feature of the model is reverse-engineering storybook curricula from Asia. Reverse-engineering helps to assure world-class content, and also gives a good start on pedagogy and pacing for classroom practicalities. Not surprisingly, the model amounts to taking the best aspects of each of the three models discussed above, which means no new funding agency or special infrastructure is required. The changes amount to how grants are written, evaluated, and implemented. Specifically, changes are needed in how authors are selected, the writing environment, and textbook composition.

Author Teams. Instead of principal investigators of grants self-selecting themselves to be authors, they should consider limiting their role to what they do best: write grants, setup the writing environment, oversee the grant, and sit as the board of advisors (quality control). In writing the grant, they should spell out how authors of the curriculum will be selected. Hopefully it will be a process which identifies the very best authors the U.S. has to offer, keeping the following in mind:

- authors are chosen for their competence, not because they have the "right" college degrees;
- the selection process is transparent and open; and
- authors are allowed to include their regular illustrators, editors, etc. as staff.

Authors should be selected for their particular talent. For example, an award winning author of children's books is preferable to a "Professor of Children's Literature." (Oh, if only Dr. Seuss was still around...) Such an author is needed for their expertise in age-appropriate writing, illustrations, layout, etc. A well-respected mathematician-author is also needed; he or she is absolutely essential for building a retrospective re-cogitative plot over eight grades, and for ensuring that the exposition is precise without being overly-technical. While no formal structure exists (like awards) to select such an author, a survey of mathematicians would probably converge on a few mathematicians-authors who are admired for exactly this ability. Finally, a (pragmatic) master teacher with impeccable writing skills is needed to assist with lesson plans and to set the pace of the textbooks. This is a teacher who has hopefully taught mathematics in most elementary grades several times. Each author comes with a particular skill that helps keep the other authors in check—preventing the curriculum from becoming overly technical, too slow-paced, too far off-task, etc. The number of authors should be kept to a minimum of four or less, but supported by a large team of teachers and staff who offer suggestions, try out the material, and give feedback. All textbook writing should be the work of the authors, not the staff.

Writing Environment and Process. The principal investigators' main operational responsibilities in this model are to create a good work environment for the authors, where the pressure is on writing well instead of on producing a product quickly. Principal investigators should ensure authors:

- reverse-engineer Asian curricula,
- write one grade at a time,
- write the textbook based on classroom-tested lecture notes and lesson plans,
- write with input from teachers,
- teach the material as it is written, and
- are given plenty of time.

One can envision the first two year period of the project has everyone on the core team (authors, teachers, etc.) teaching first and second grade classes in the morning and using the afternoon for creating next weeks lecture notes and lesson plans. This time is needed for some of the authors, who are possibly new to teaching elementary school, to get used to the pace and teaching issues of early grades. It also gives the core team a chance to intensely study storybook curricula like Singapore's, Hong Kong's, Japan's, etc. Only in the third year, after the lesson plans have taken shape, is the first grade textbook fleshed out and written to accompany the lesson plans and lecture notes. Every year thereafter a new textbook is written; so, for example, in the fourth year the authors are teaching grades 1-3, editing the first grade textbook, and writing the second grade textbook. In this setup, cycling through grades three at a time, the curriculum would take 11 to 12 years to write.[13]

[13] Timelines similar to this one have been used by some curricula writers, see *Everyday Math*'s website.

The idea that the authors should teach at the same time as they write is very important. If there is one thing I have learned from writing textbooks, it's that there is absolutely no substitute for classroom experience when writing. Students have a profound way of immediately illuminating faults in the text by the ways they interpret it, and the only way to see that illumination is to be there in their presence.[14] The advice goes for the support team and anyone else who has major input into the textbooks: they too should teach along side the authors. Their feedback has far more impact on authors when it has the force of 25 dumbfounded students behind it.

Textbook Composition. This is the trickiest aspect of the model to control from the standpoint of a principal investigator because one does not want to limit the authors' creativity. With that said, the following guidelines, taken directly from what can be observed in the Singapore textbooks, are meant to spur authors' ingenuity rather than stifle it. For example, the second guideline could be restated, "Brevity is the soul of wit." Authors should:

- write for the diligent student who started in first grade with the curriculum,
- limit the number of textbook pages to 200 pages per grade, but create a lot of support material,
- write with mastery as the goal, not just awareness,
- juxtapose each new topic with an old topic,
- use mathematical definitions that become more refined as students progress,
- choose a simple layout with lots of white space, two main colors, and one font,
- use cartoon kids to announce key ideas, definitions, and facts, and
- choose a few (<10) models and use them consistently throughout grades 1-8.

Most of the advice above is self-explanatory after reading the first part of this article, but I want to comment further on the first two guidelines. It is well known that many parents and students in the U.S. move between different school districts often. A good starting point for the solution to this bouncing-problem is a national curriculum, but such a solution (or even its shadow solutions) is unlikely to materialize soon. Many curriculum designers have attempted to develop their curriculum around this problem, writing for the student who moves often rather than for the stable one who stays put. In this model the authors are writing for the student who stays, i.e., the one who can benefit the most from a coherent curriculum. That does not mean the moving student is ignored, not at all! Remediation programs are always needed to give students who are behind for whatever reason a chance to catch up and rejoin the curriculum. Writing for the diligent student who stays also has a double benefit for advanced students: supporting workbooks written parallel to the main storyline can inspire bright minds by having them to delve deeper into the same material.

Award Driven Model. The grant driven model above covers textbooks created in university settings, but how can the U.S. government encourage for-profit publishing companies to design story-like curricula? Nothing right now encourages publishers to write story-like curricula—they are wedded to the current system because it is known to produce profitable material and because it is also what teachers and administrators have come to expect when selecting curricula for their schools.

It is possible to (positively) change the way for-profit publishers create curricula. An instructive example is a highly successful program created in the late 1980's to encourage appliance manufacturers to design high-efficiency refrigerators. Twenty four utility companies joined forces to fund the ``Super Efficient Refrigerator Program,'' a $30 million prize awarded to the appliance company that designed the best refrigerator, based upon set of energy saving performance criteria. While $30 million seems like a large sum of money, it was only meant to help the winning company defray some of the startup costs in bringing the new product to market.

[14] Somewhat unbelievingly, I have observed other authors write without teaching; something I liken to writing in a cave.

Thirteen major companies, Whirlpool, GE, Maytag, etc., competed fiercely with each other to design the refrigerator with the highest energy efficiency. They did this partly for the prestige of winning the award, but also for the positive publicity and marketing boost that would result from winning. In the end, all of the major appliance companies introduced innovative new high-efficiency refrigerators, and all of them marketed and sold those products even though only one company won the prize. Today the U.S. continues to enjoy the long-term benefits of reduced daily energy usage/costs because of the award program. It has also spurned a new way of marketing appliances to consumers based upon energy and water conservation, which has influenced consumers' criteria for buying refrigerators.

The same can be done for textbook publishers: create an award that could adequately cover the costs of bringing a new elementary curriculum into the market and use the criteria discussed in the grant driven model above as a basis for determining the winner. The hope is that a similar innovative spirit could take hold in publishing companies, and in the end, not only would school districts have several story-like curricula to choose from, but the teachers' perceptions of what mathematics curricula should look like might change over time as publishers revamped their marketing strategies to highlight the story-like aspects of their new curricula.

Conclusion

There is a wealth of important considerations that were not covered (in depth) in this article—pedagogy, teacher training, parental involvement, and testing just to name a few. Some curriculum authors have dealt with these issues by writing bold, new curricula with new content and new pedagogy. The emphasis in this paper is not so much on new content or new pedagogy, but simply on the craftsmanship of the actual writing. Shakespeare took many of the ideas for his plays from history and/or old plays. It was the way that he crafted those existing ideas into engaging stories that turned them into masterpieces.

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View Mathematics Curriculum Research Papers on Academia.edu for free.Â The cognitive domain in mathematics, defined as thinking and understanding in the process of learning mathematics, is a main focus of curricula in many countries. This study explores breadth and depth of understanding as addressed in more. The cognitive domain in mathematics, defined as thinking and understanding in the process of learning mathematics, is a main focus of curricula in many countries. This study explores breadth and depth of understanding as addressed in mathematics textbooks certified as aligned to Israeli national mathematics curricula. I hope that the Australian curriculum is not so driven by such assessment considerations. In considering assessment, Sullivan points out that the PISA 2009 Australian data show that, despite central initiatives, the attainment gap between children from high and low SES home backgrounds seems to be widening.Â We need to remember that school mathematics has a â€˜social perspectiveâ€™ in and of itself and that some students will find meaning in contexts that are purely mathematical.Â In the final section of this research review, Sullivan summarises the implications for teacher education and professional development. As he indicates, there is still much work that needs to be done to improve mathematics teaching and learning. MOE develops the. national mathematics curriculum and oversees its implementation in all schools, while the NIE is involved in teacher preparation and development and also research. in mathematics education. This report comprises two sections: the ï¬ rst descri bes the. education system and school mathematics curricula while the second provides. relevant information on teacher preparation and development and mathematics. education research in Singapore.Â and the Singapore Mathematical Society (SMS) are also actively engaged in the PD. of teachers. They hold relevant annual meetings, seminars and conferences for. In Singapore, mathematics education is guided by a framework covering the development of five domains â€” concepts, skills, processes, attitudes and metacognition, where mathematical problem-solving is the central focus, and this framework sets the direction for teaching, learning and assessment in all schools in Singapore from primary to the â€˜Aâ€™ levels. The different levels of syllabuses are classified according to topics and subtopics in arithmetic, algebra, geometry, measurement, trigonometry, statistics, probability and calculus.Â Very often, mathematics educators do not distinctively separate between concepts and facts when delivering lessons in the classroom. Past research on the intended curriculum for mathematics and reading is briefly reviewed. The purpose of the study and the specific activities this project entails are also clearly stated. Section 2 gives a detailed description of how the intended reading and mathematics curricula are compiled, coded and compared. The extent and diversity of data coverage within regions are reviewed.Â Several observations and possible implications for improving learning assessments in the developing world are also discussed in this section. Section 5 proposes future activities that can help build on the existing ICATA archive to improve, expand and make the archive more policy-relevant by addressing language-related questions.